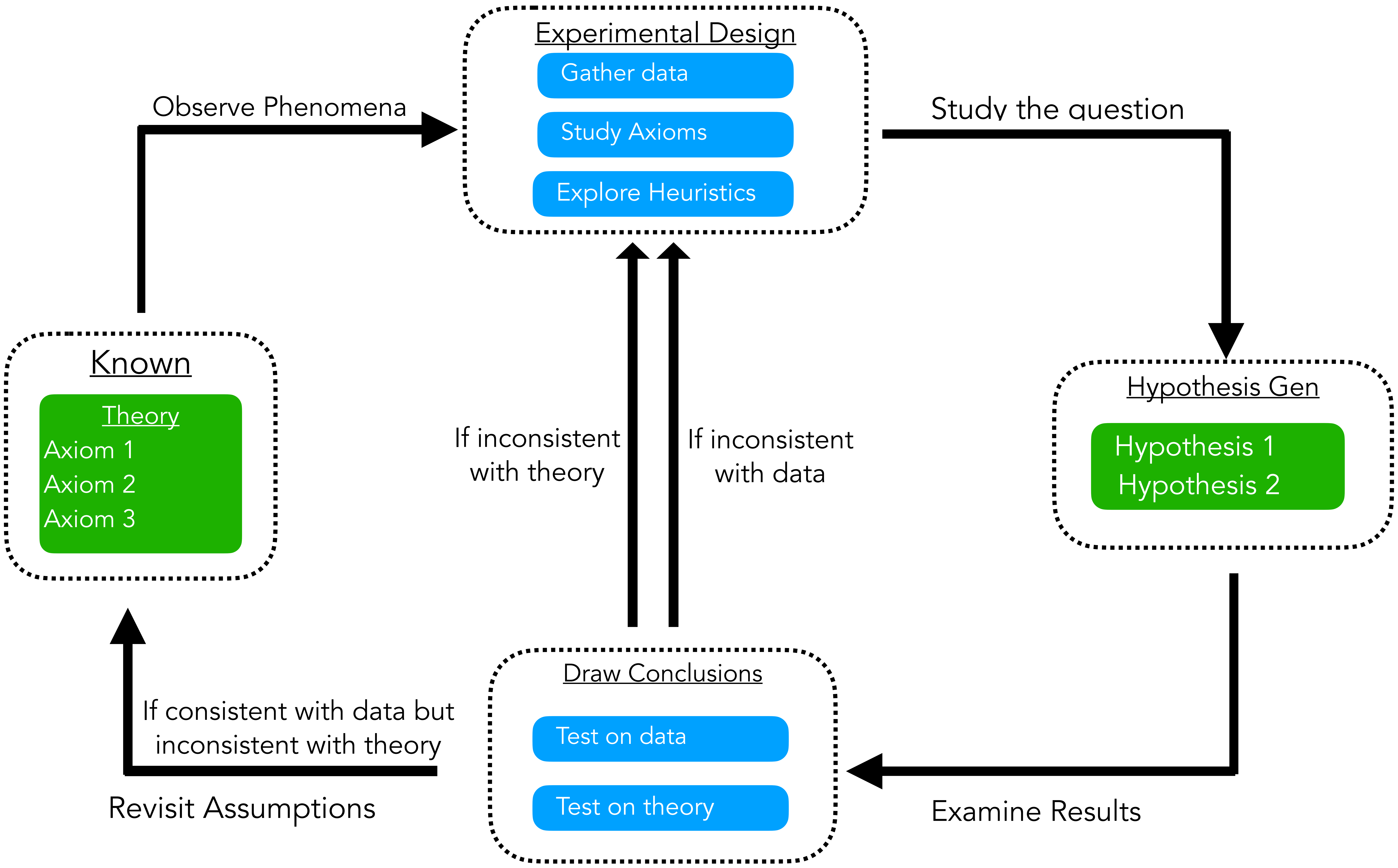
Computational algebraic methods for abductively inferring axioms to explain a phenomenon.

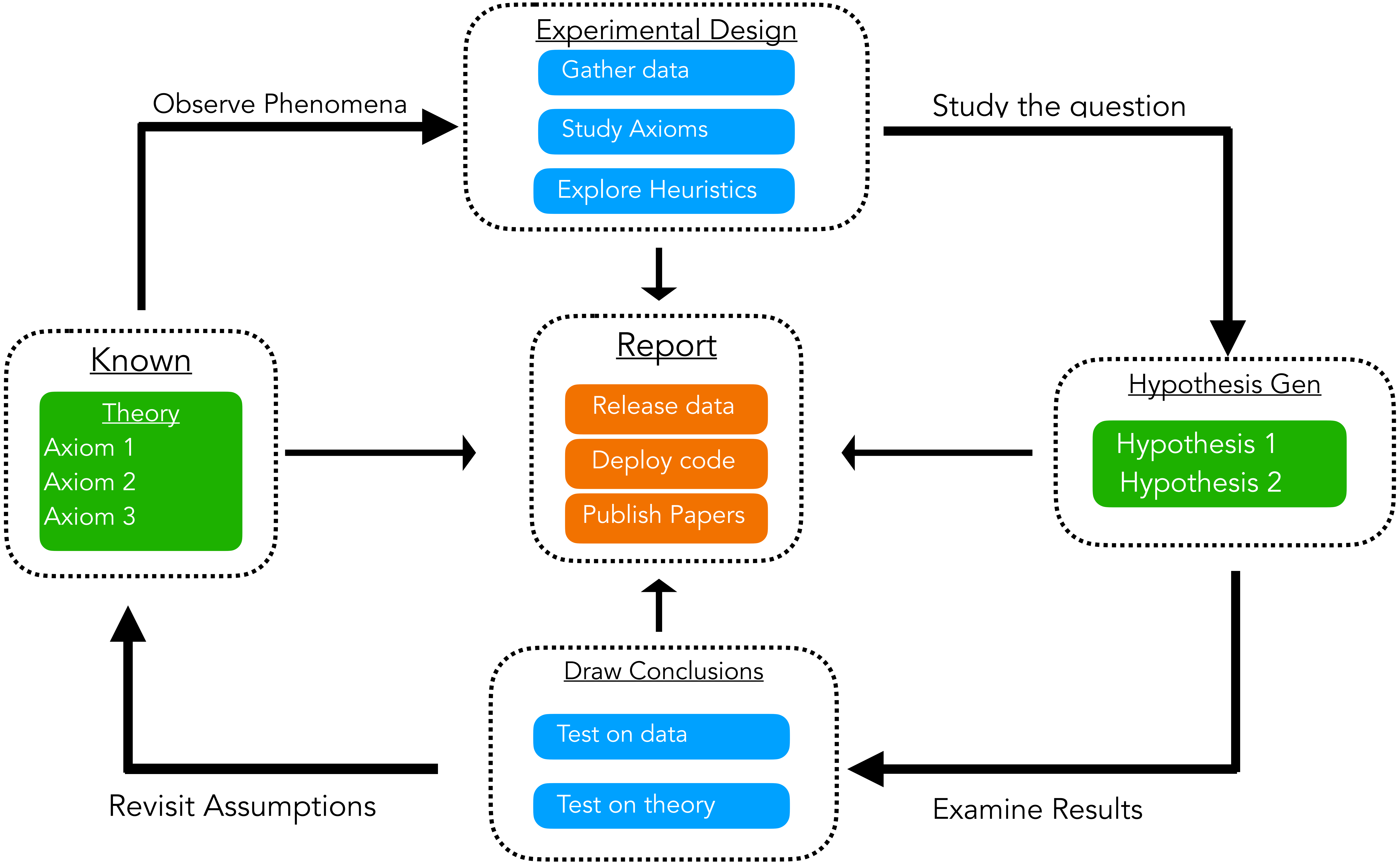
Karan Srivastava (UW Madison, IBM), Sanjeeb Dash (IBM), Barry Trager (IBM), Ryan Cory-Wright (Imperial College) , Lior Horesh (IBM)

Alan Turing Institute, Machine Learning and Dynamical Systems Seminar Talk

The Scientific Method



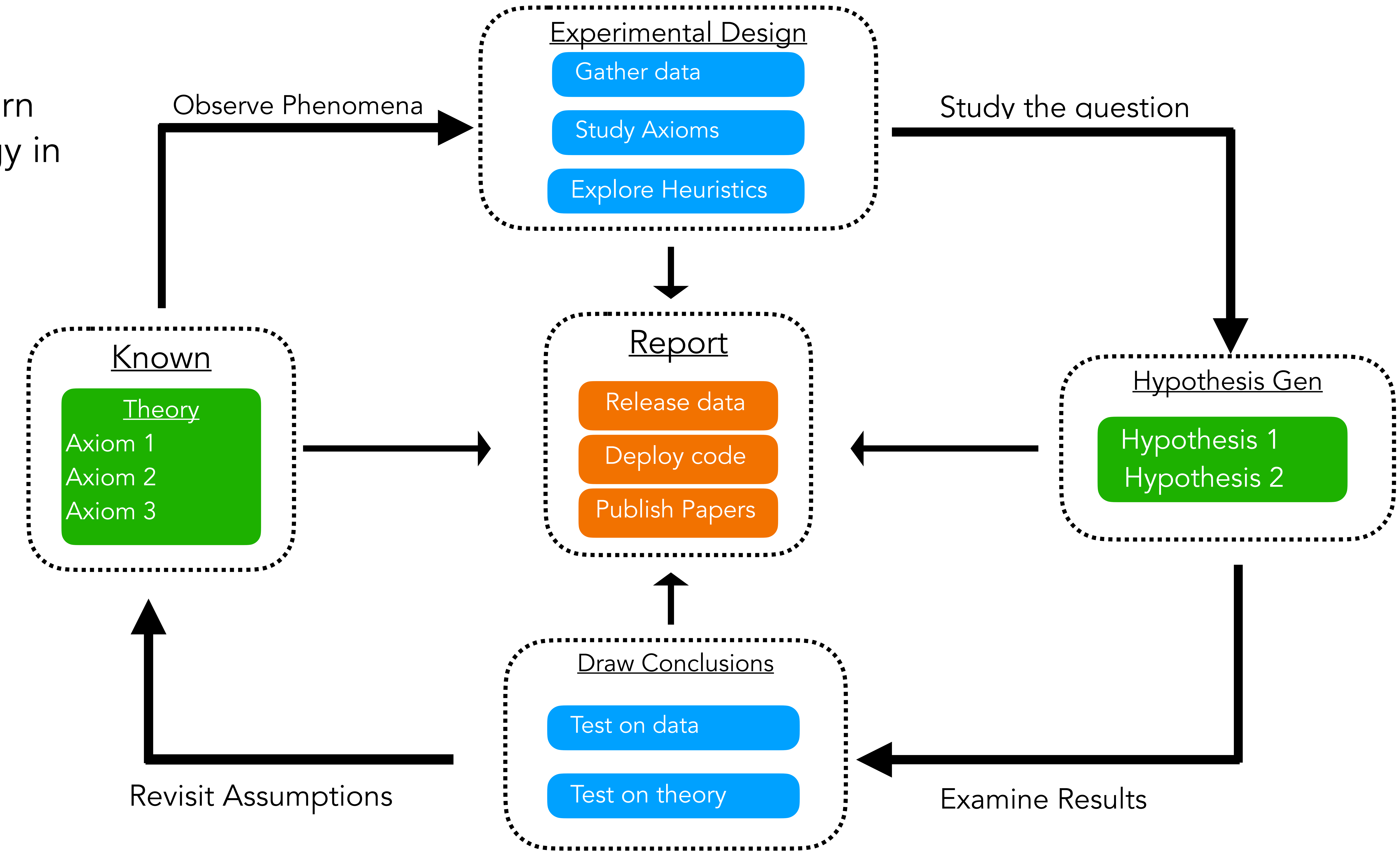
The Scientific Method



The Scientific Method

Key Question:

How have we utilized modern mathematics and technology in this process?



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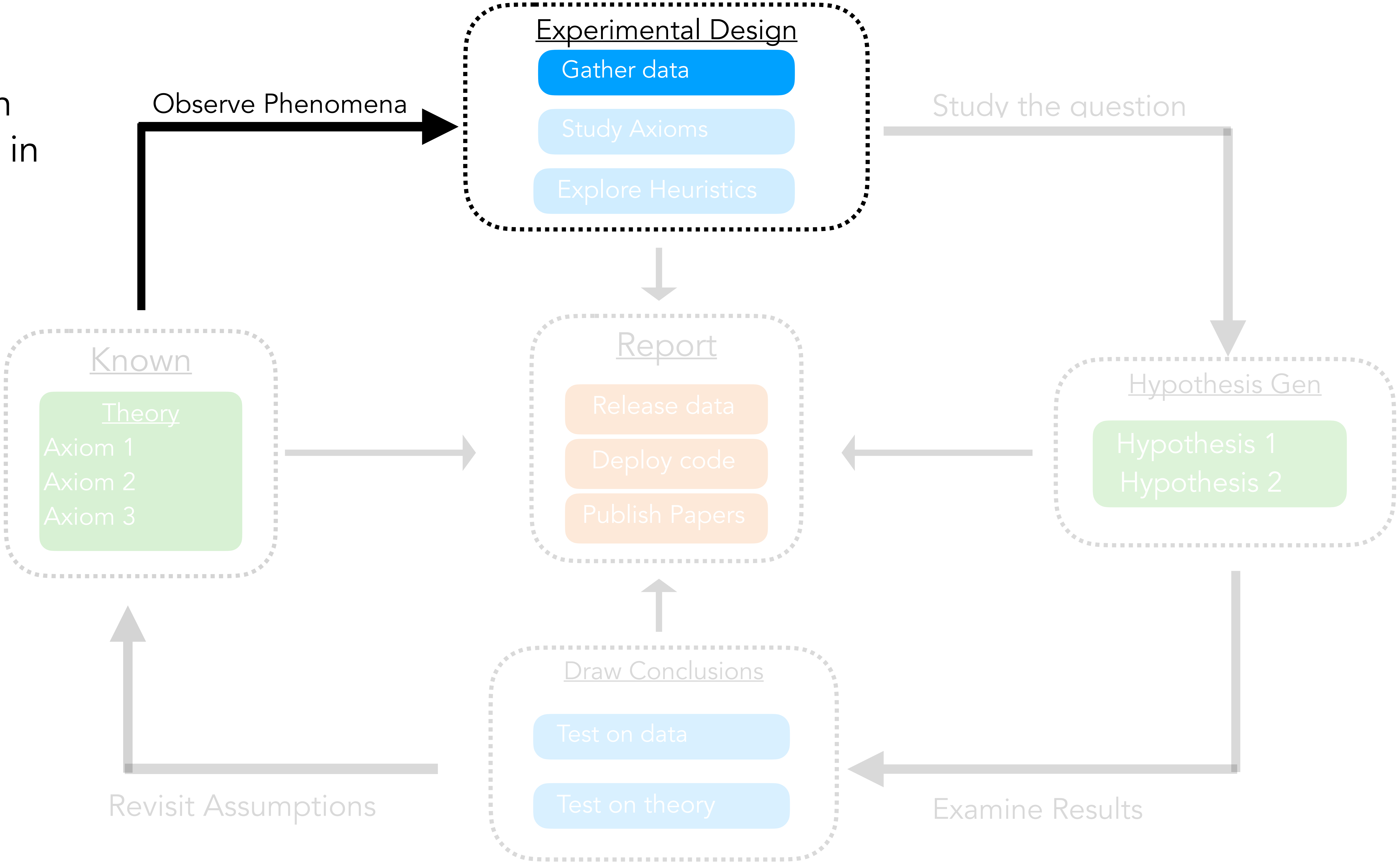
Observing Phenomena

Hardware and technology developed for extending our observational capacity and ability to gather data.



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Known

Theory

Axiom 1

Axiom 2

Axiom 3

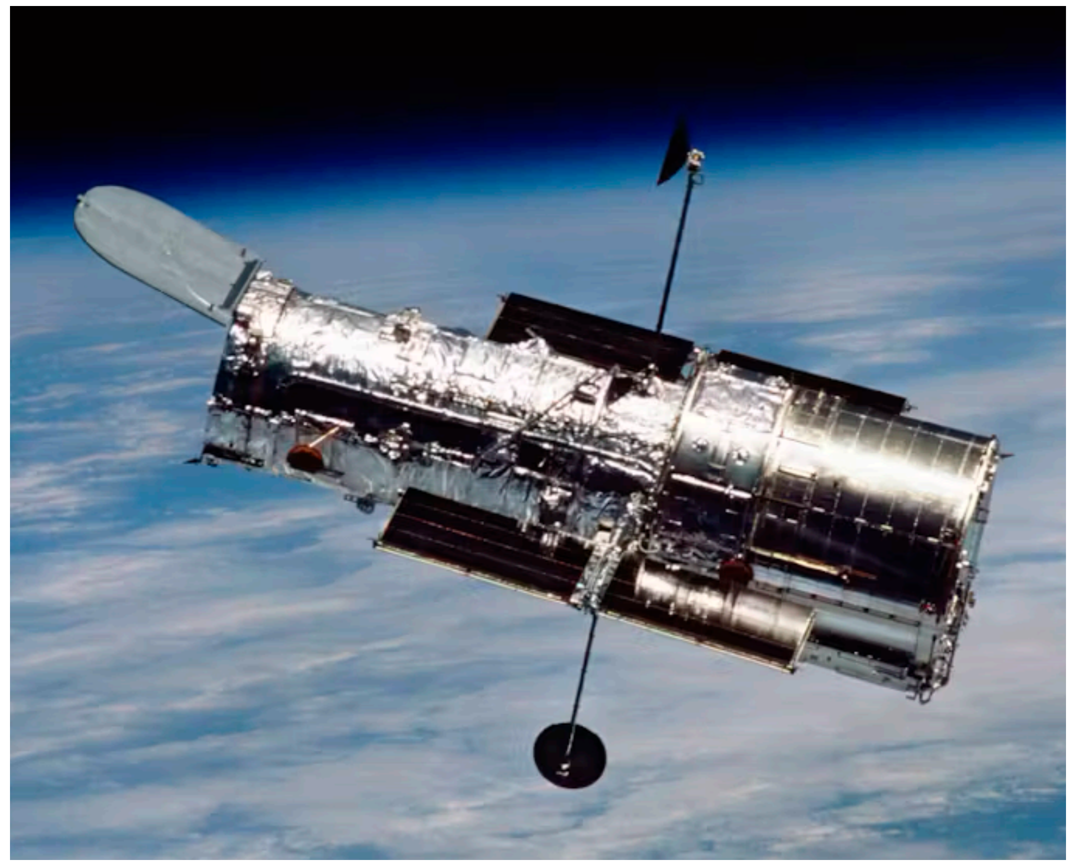
Observe Phenomena

Experimental Design

Gather data

Study Axioms

Explore Heuristics



Hubble Telescope generates hundreds of gigabytes of data per month.

Large Hadron Collider generates data for the study of particle physics.

The Scientific Method

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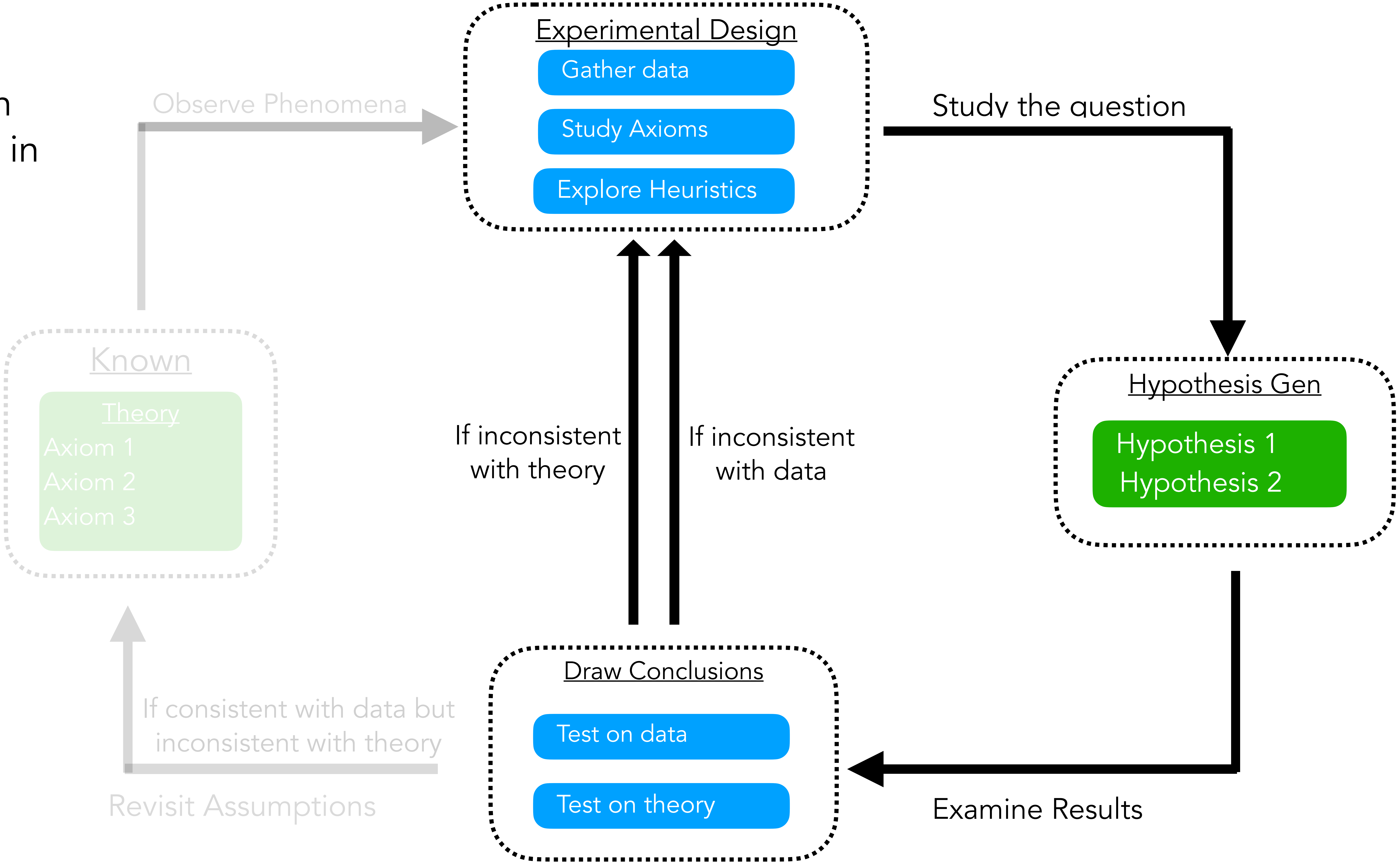
Hypothesis Generation and testing

This has been the topic of the “Scientific Discovery with Statistical, Symbolic, and Gen-AI” series of talks in this seminar.



The Scientific Method

Traditional Discovery

Discovering from first principles and testing on data.

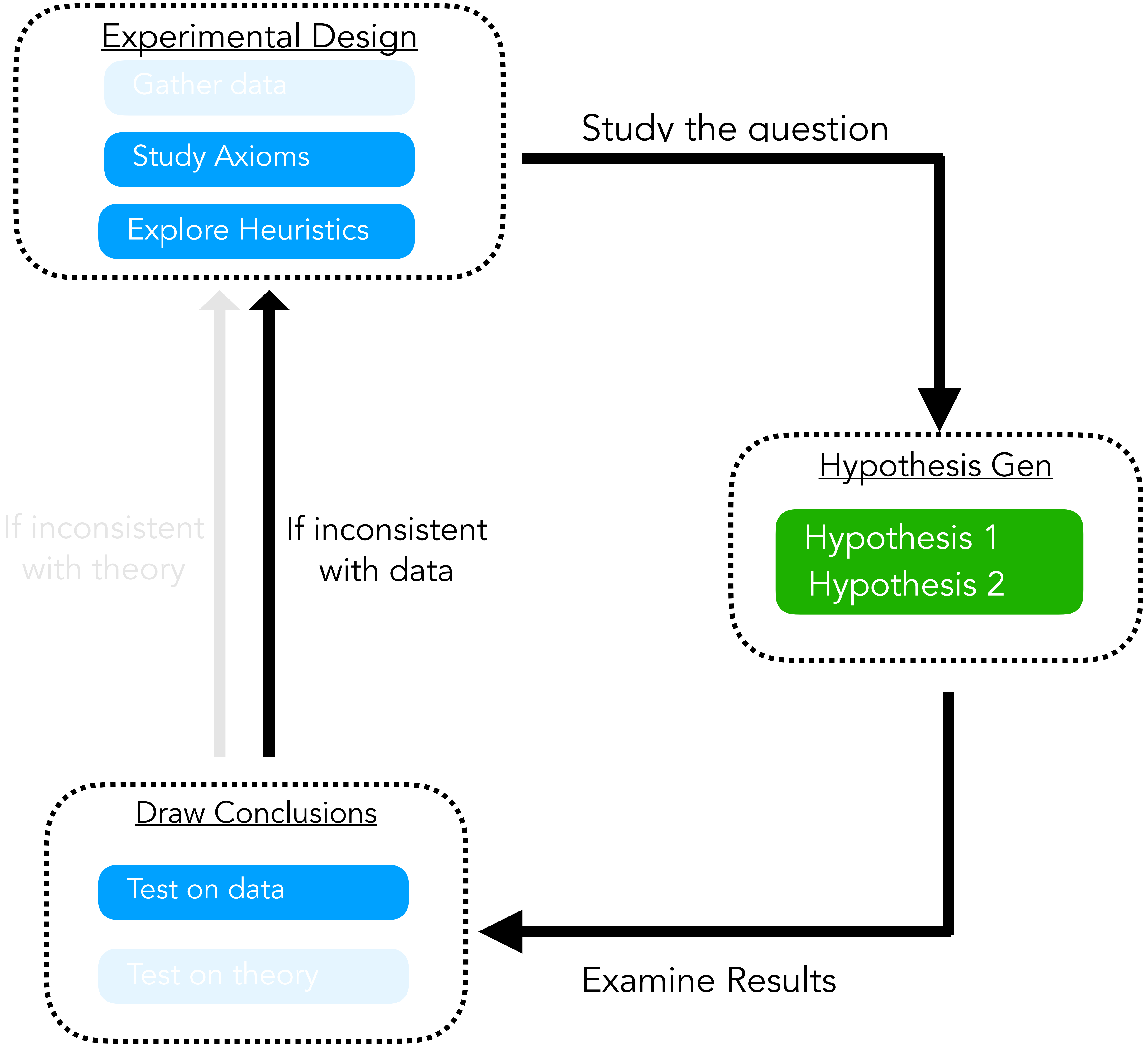
Strengths:

1. We (generally) find encodings of phenomena that are explainable from theory.

2. Proposed hypotheses are (generally) quick to test on data (if it exists).

Limitations:

Coming up with new theories is challenging to do manually.



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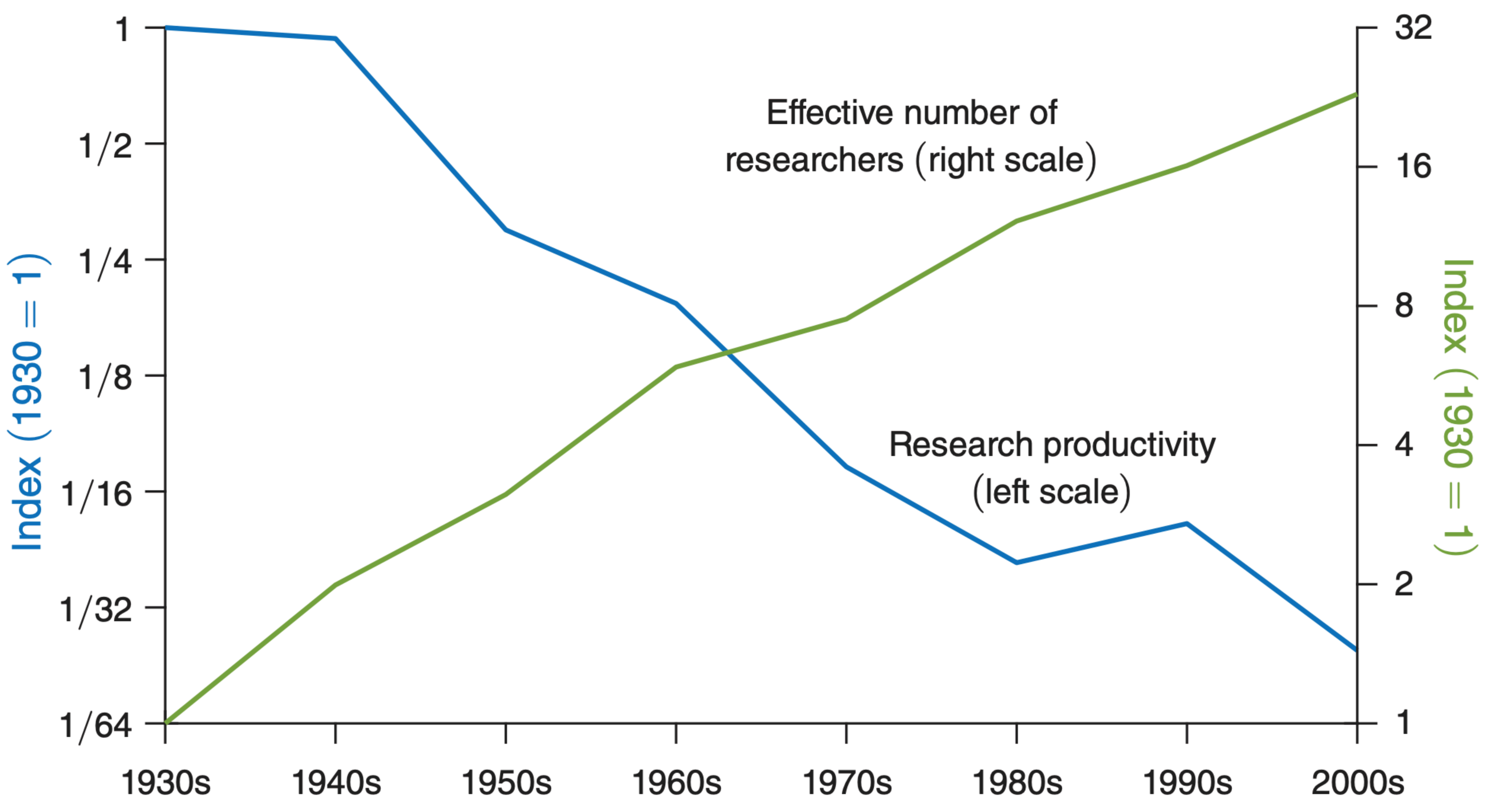
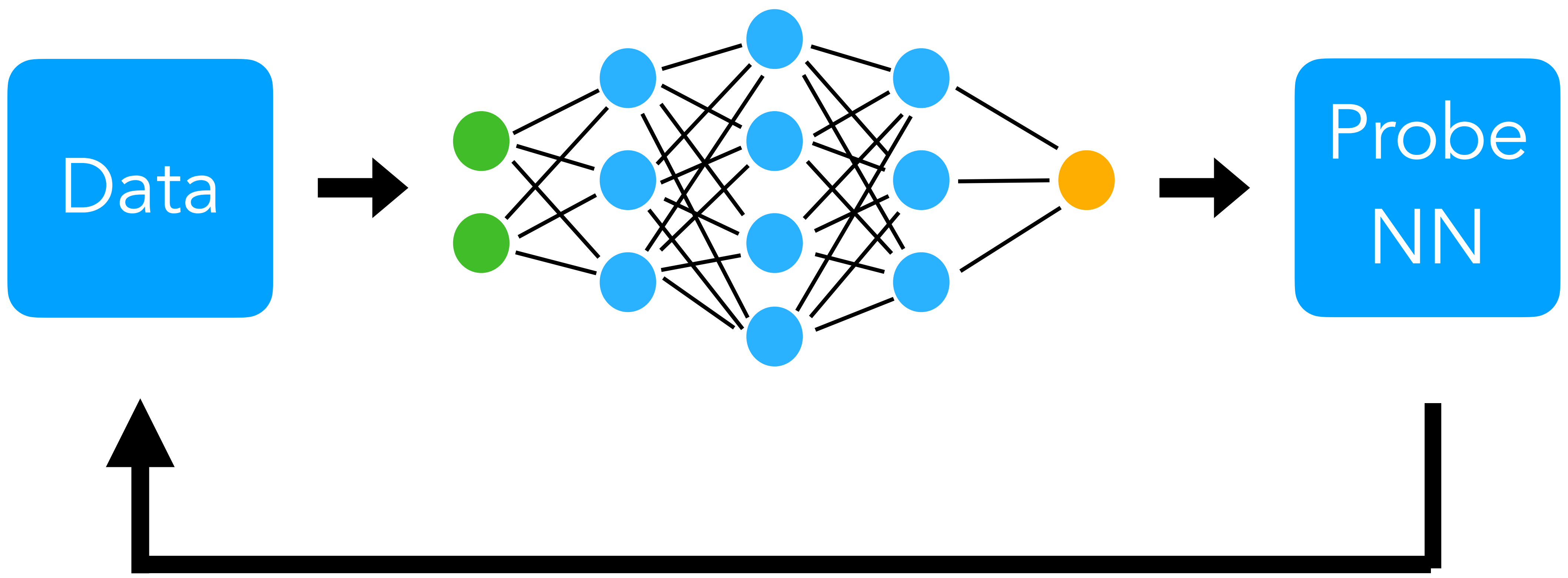


"The number of researchers required today to achieve the famous doubling of computer chip density is more than 18 times larger than the number required in the early 1970s."
Bloom et al., Are Good Ideas Getting Harder to Find?, American Economic Review, 2020

The Scientific Method

Data-Driven Methods

Discovering from data (E.g AI Feynman, Udrescu and Tegmark 2020)

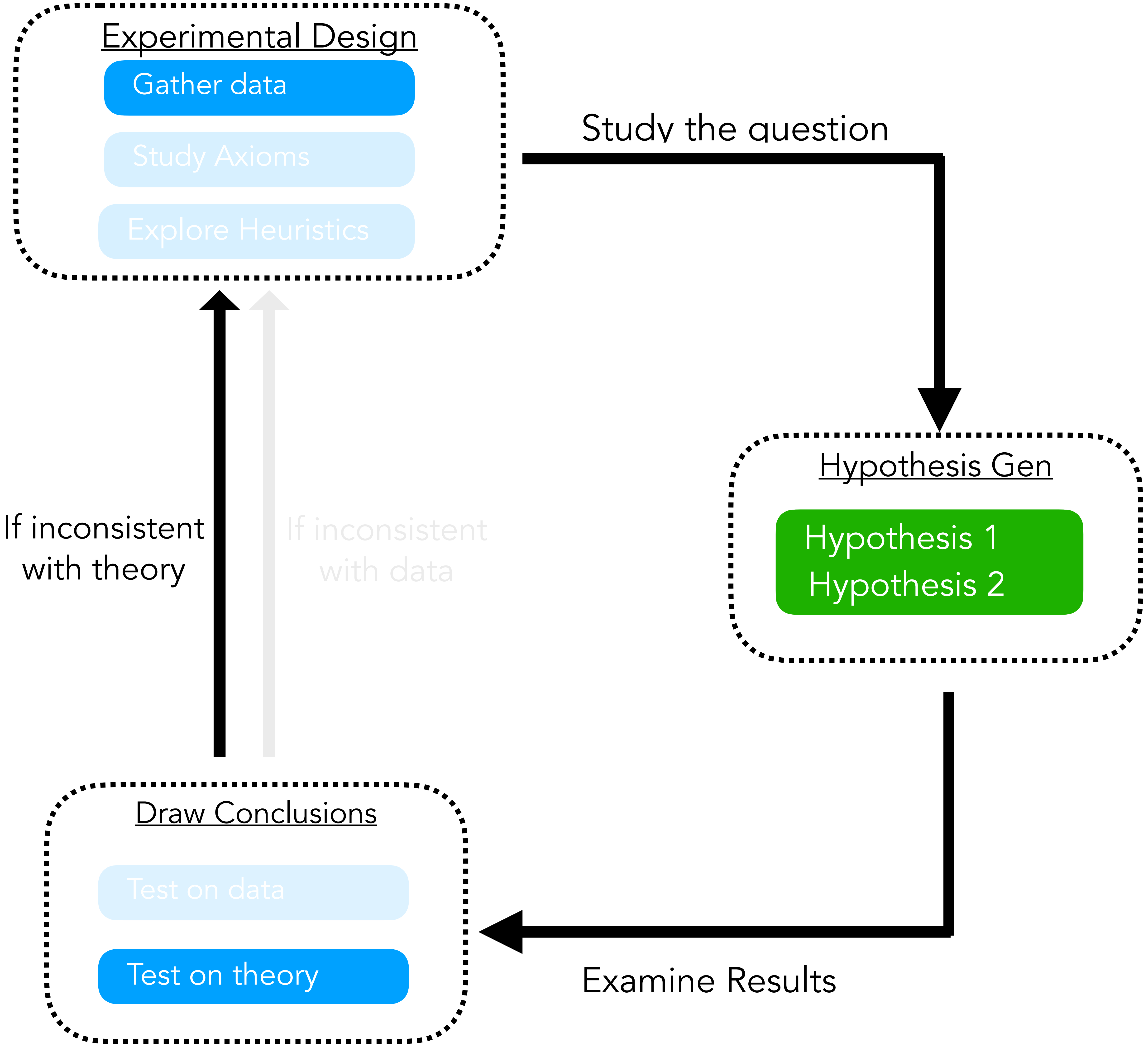


Strengths:

1. Effective when large datasets are available

Train Oracle

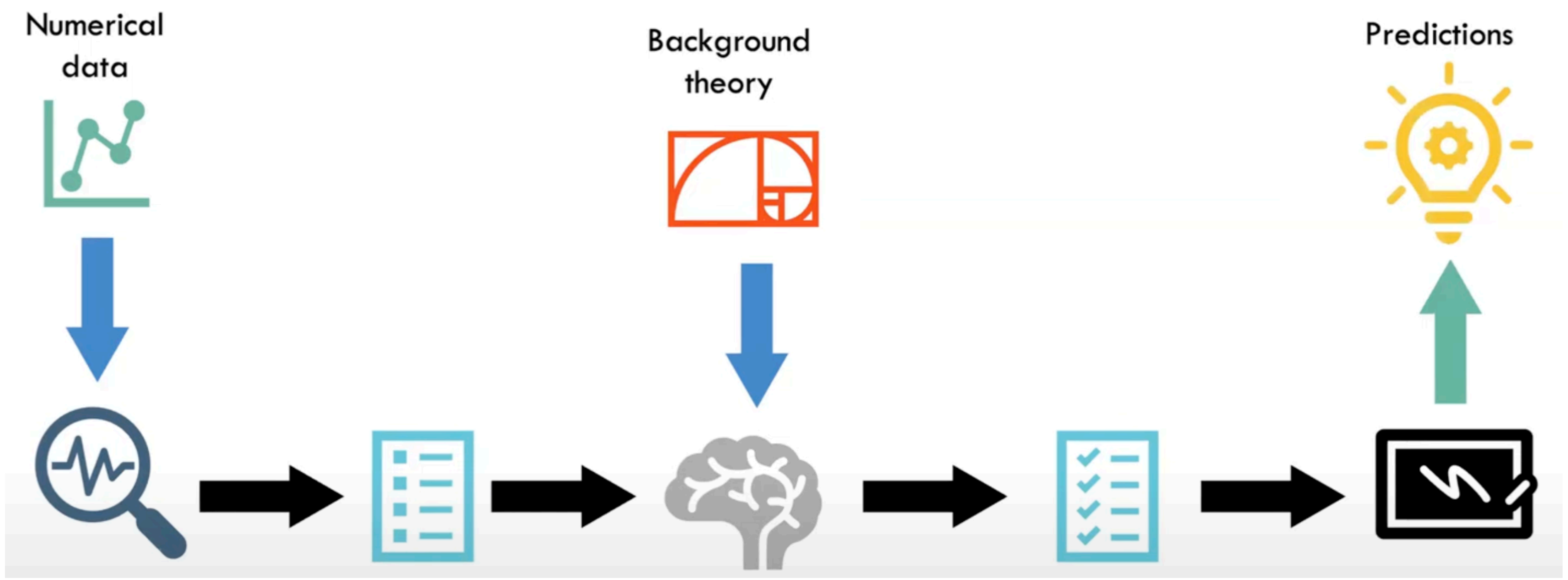
2. Little to no domain knowledge required.



The Scientific Method

Data and Background Theory Methods

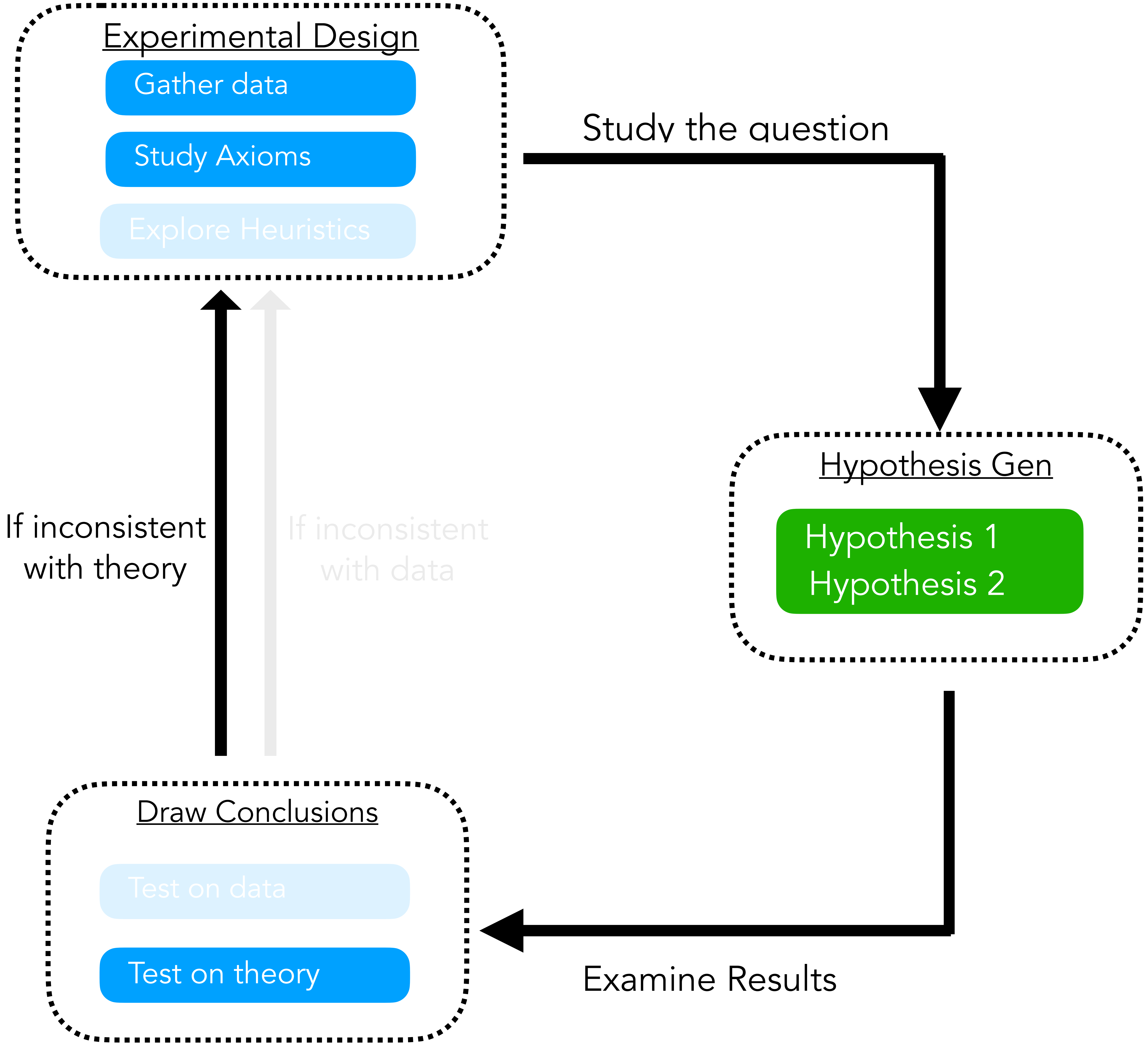
AI Descartes [Christina Cornelio, et al. Nature Comms, 2023]



Fit to data, then test on background theory

Strengths:

1. Can work with small datasets.
2. Recover a certificate of derivability from axioms along with hypothesis.



The Scientific Method

Unified Methods

AI Hilbert [Ryan Cory-Wright, et al. Nature Comms, 2024]

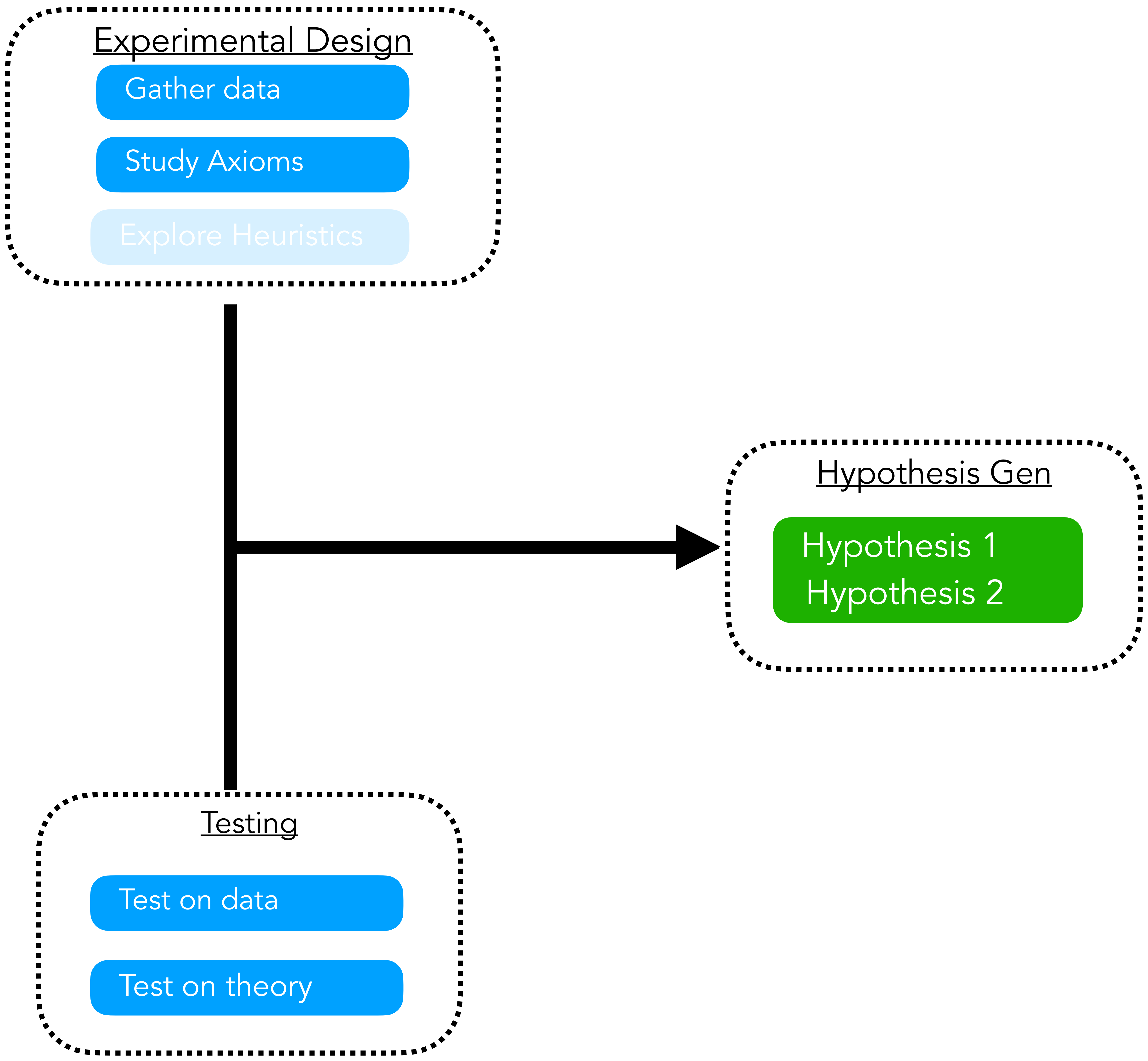
$$\min_{q \in \mathbb{R}_{n,d}[\mathbf{x}]} \sum_{\mathbf{x_i} \in \text{data}} q(\mathbf{x_i}) + \lambda d(q(\mathbf{x}), A)$$

Fit to both theory and background data simultaneously using polynomial optimization

Strengths:

1. Simultaneously fit to data along with generating certificates of derivability

2. Discovered solutions are provably optimal



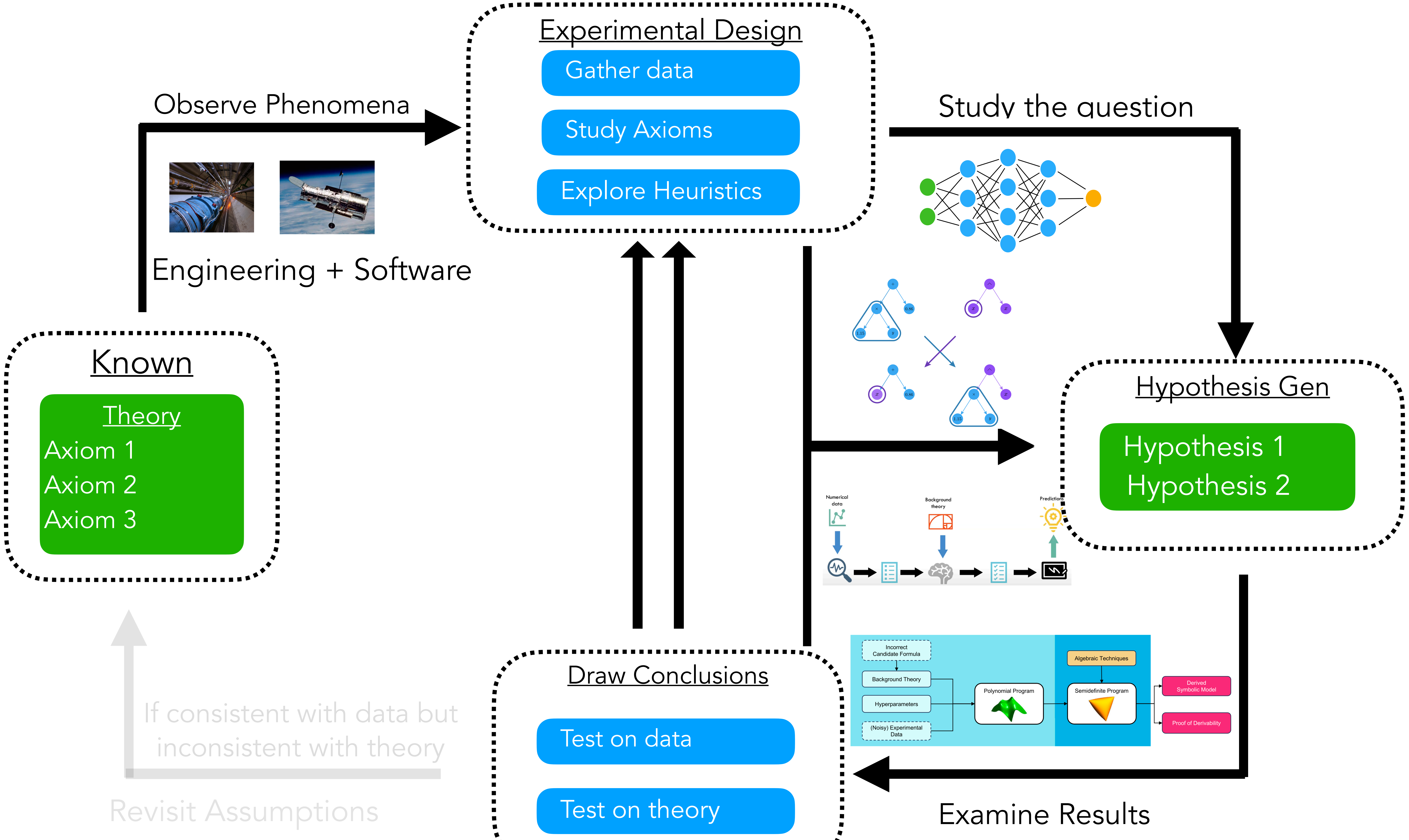
The Scientific Method

Key Question:

How have we utilized modern mathematics and technology in this process?

Answer:

For most of the modern scientific process, we get a lot of mileage from computational tools and modern technology.



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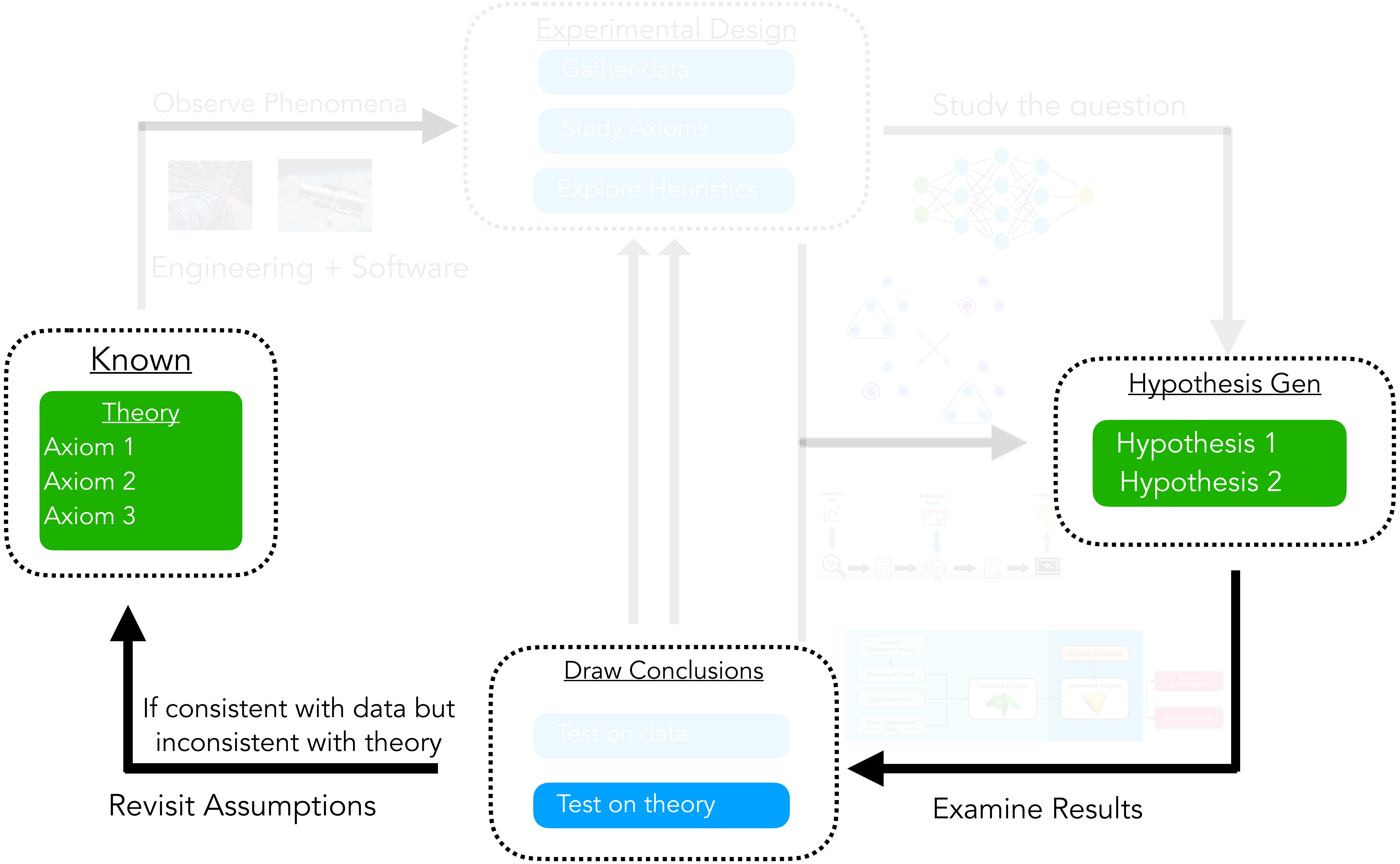
Answer:

For most of the modern scientific process, we get a lot of mileage from computational tools and modern technology.

But what about using our new hypotheses to re-examine the theory that we have?



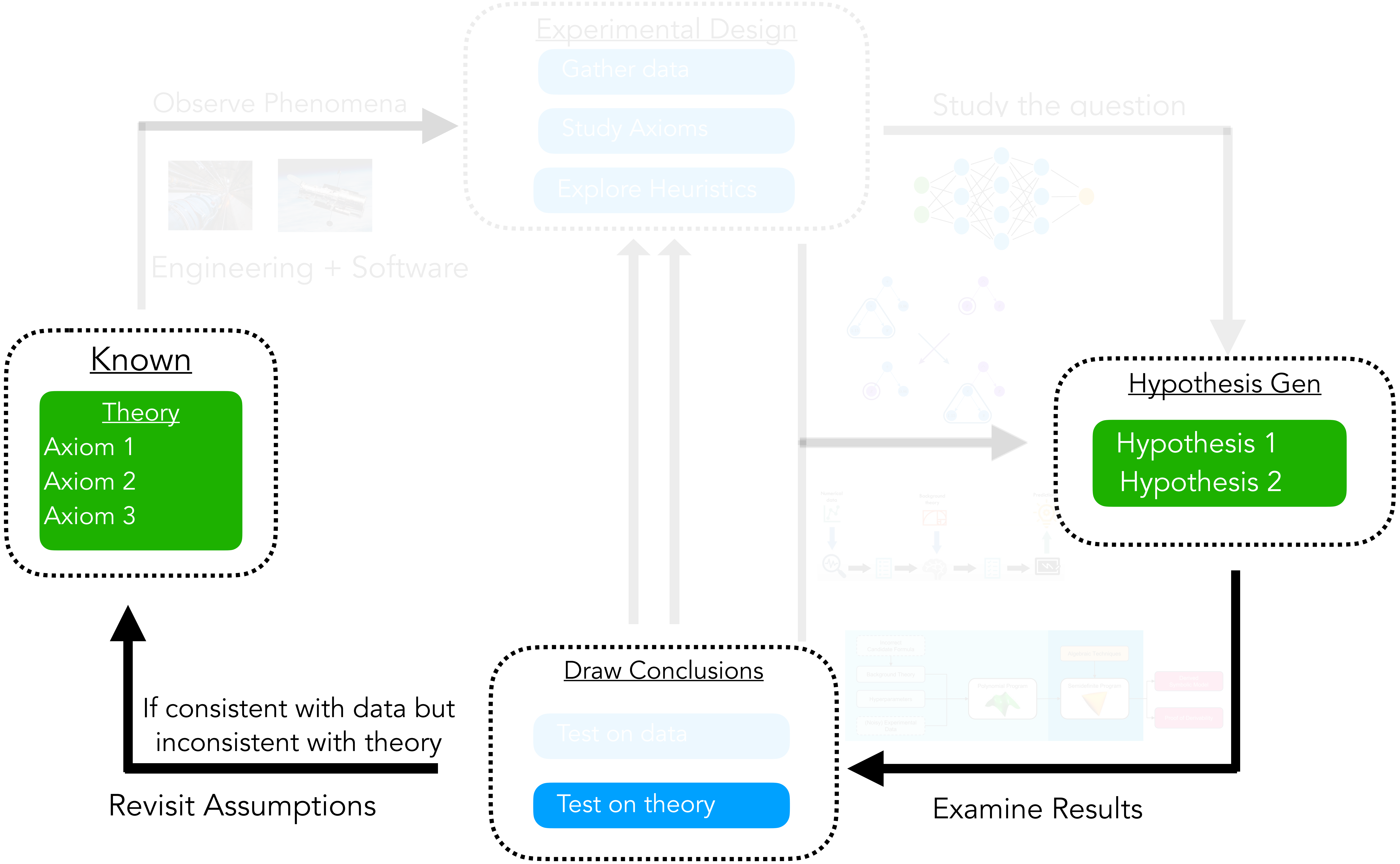
The Scientific Method

Revisiting Assumptions:

The problem with data driven systems is that they can generate formulae that are difficult to interpret or are not consistent with theory.

The data+background theory methods generate formulae that are derivable from known theory.

But often times theories can be incorrect or incomplete!



The Scientific Method

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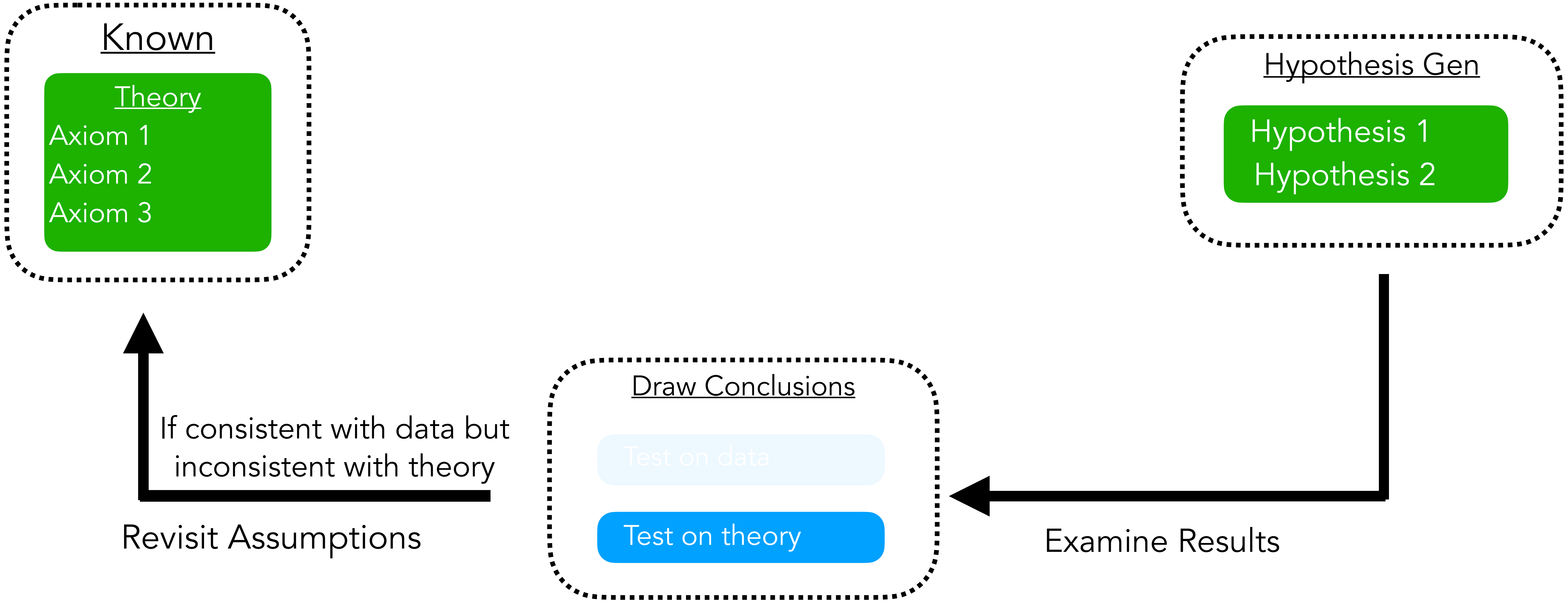
Examples:

1. Cosmology - e.g Lithium Abundance
2. Historic example: Perihelion Precession of Mercury



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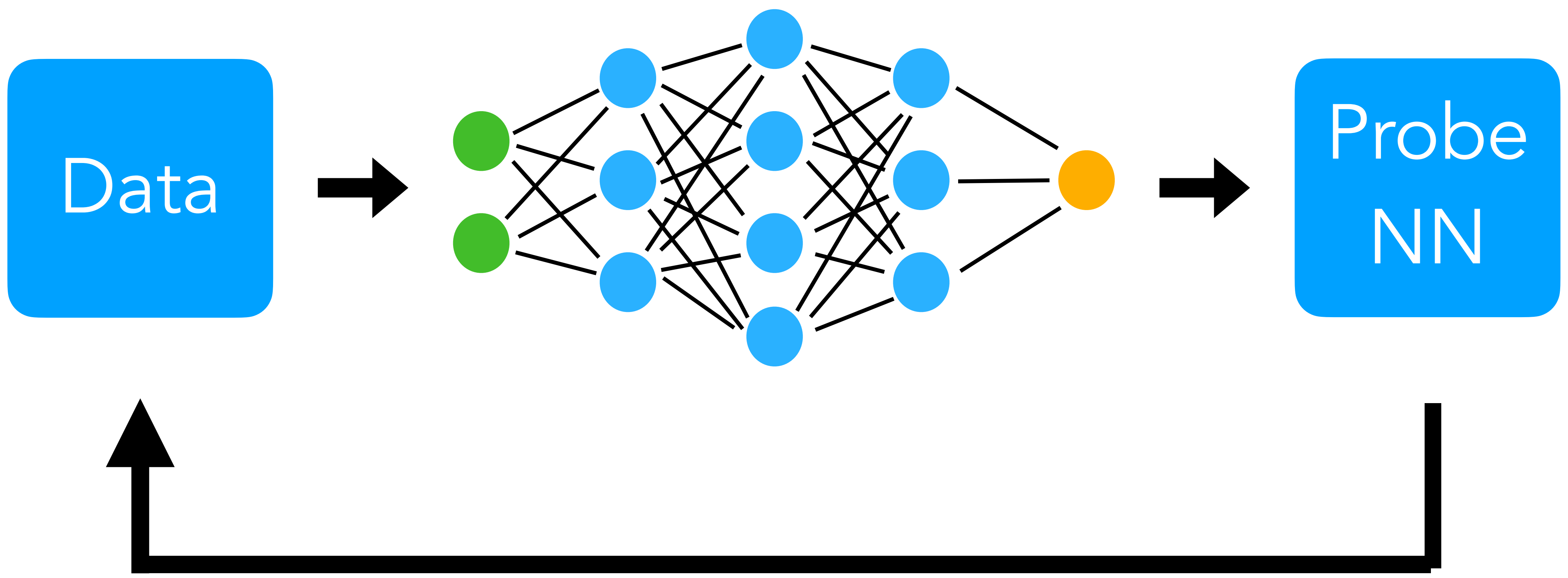
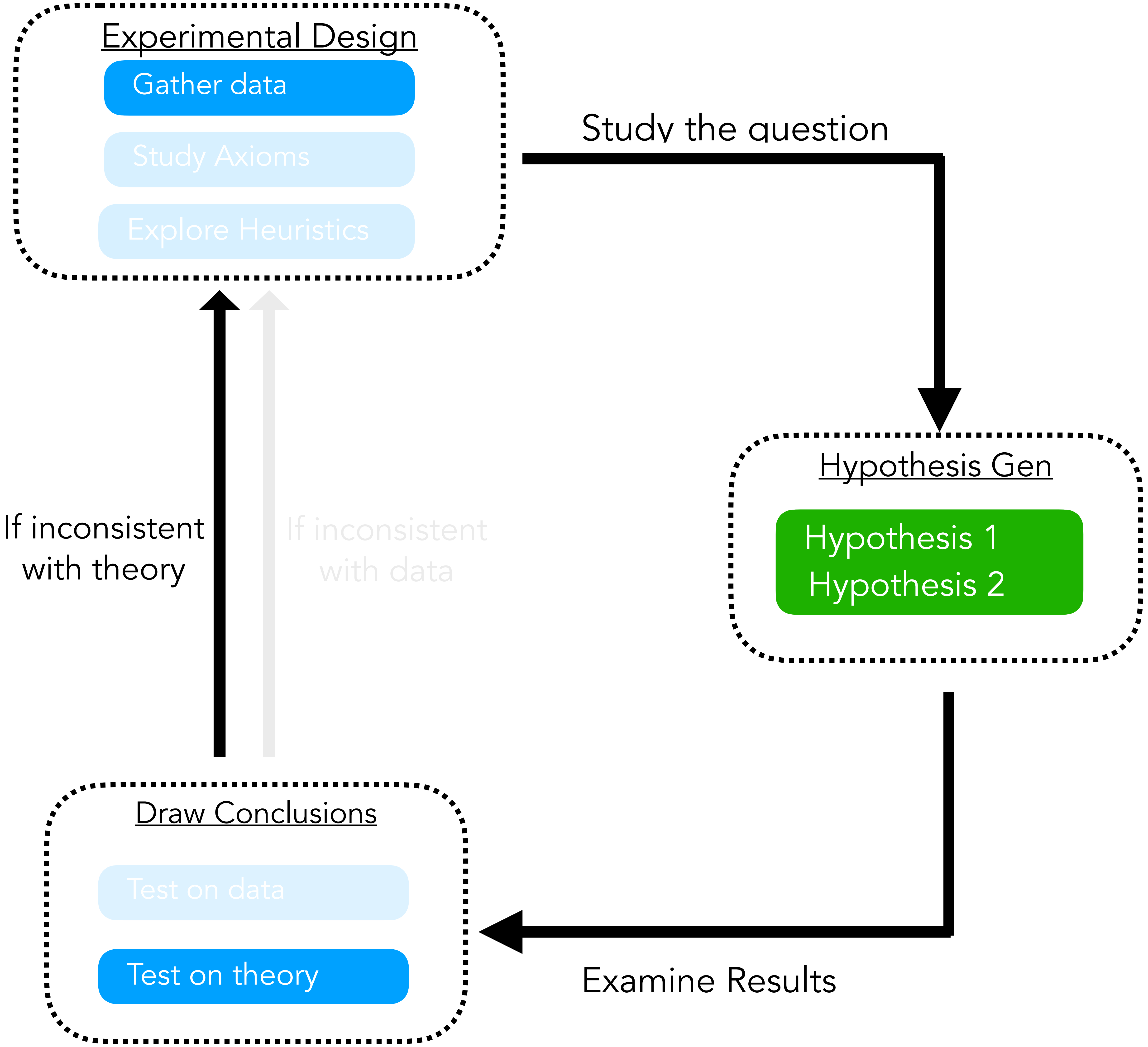


Limitations:

May fail to be consistent with theory. Less interpretable.



The Scientific Method

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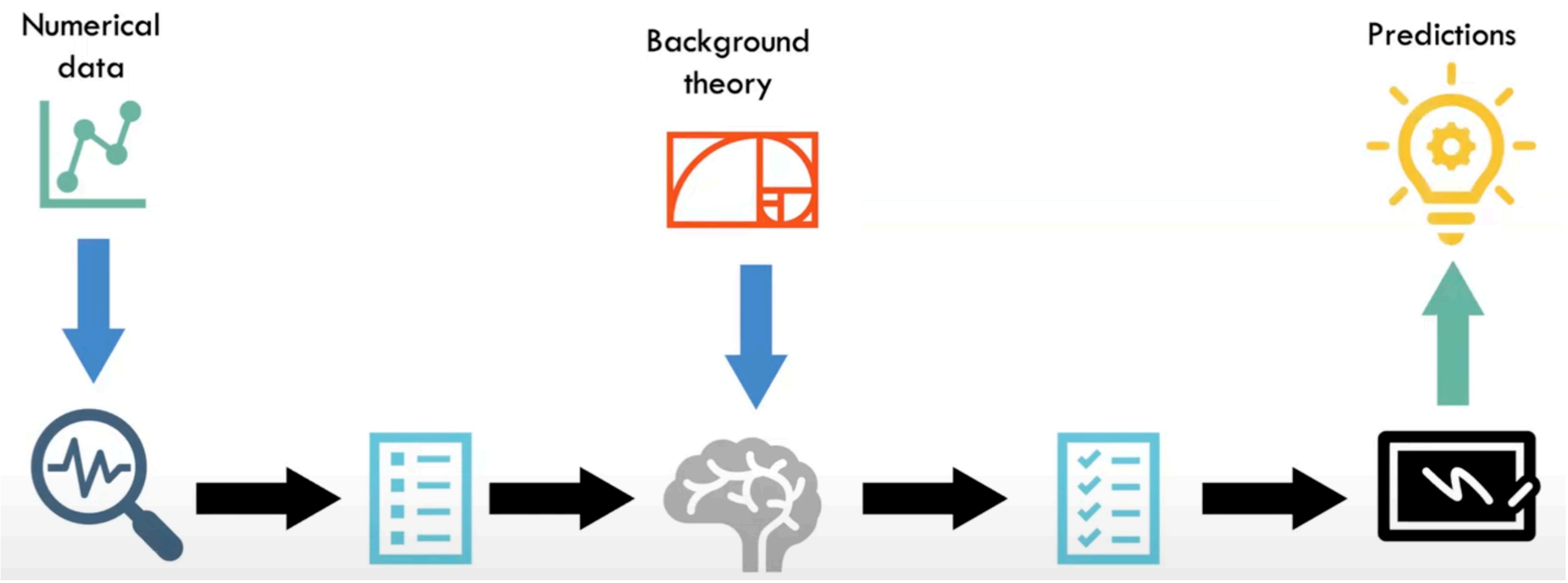
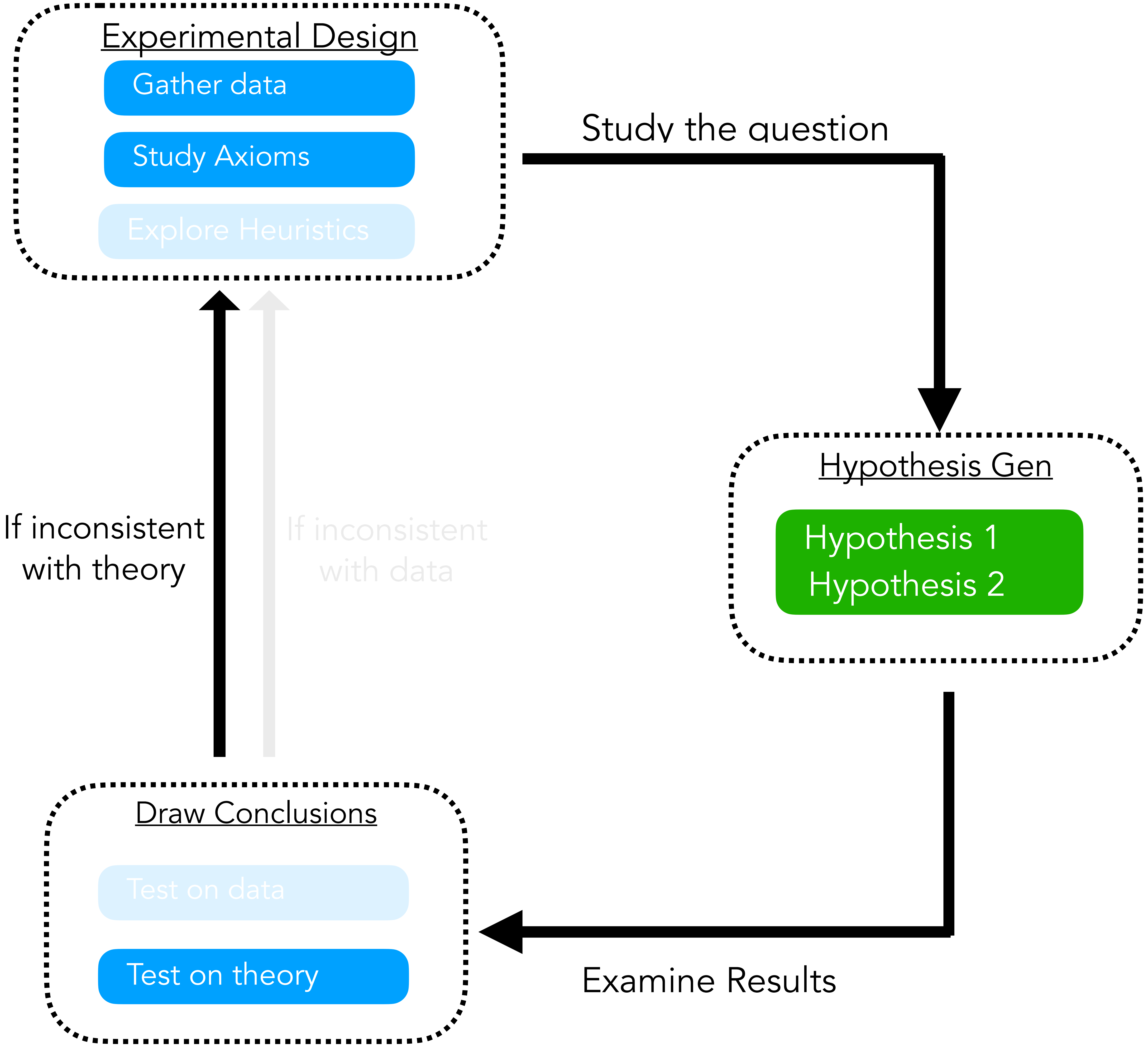


Limitations:

We primarily use background theory for verification, not search. Verification only works when the **background theory is sufficient.**



The Scientific Method

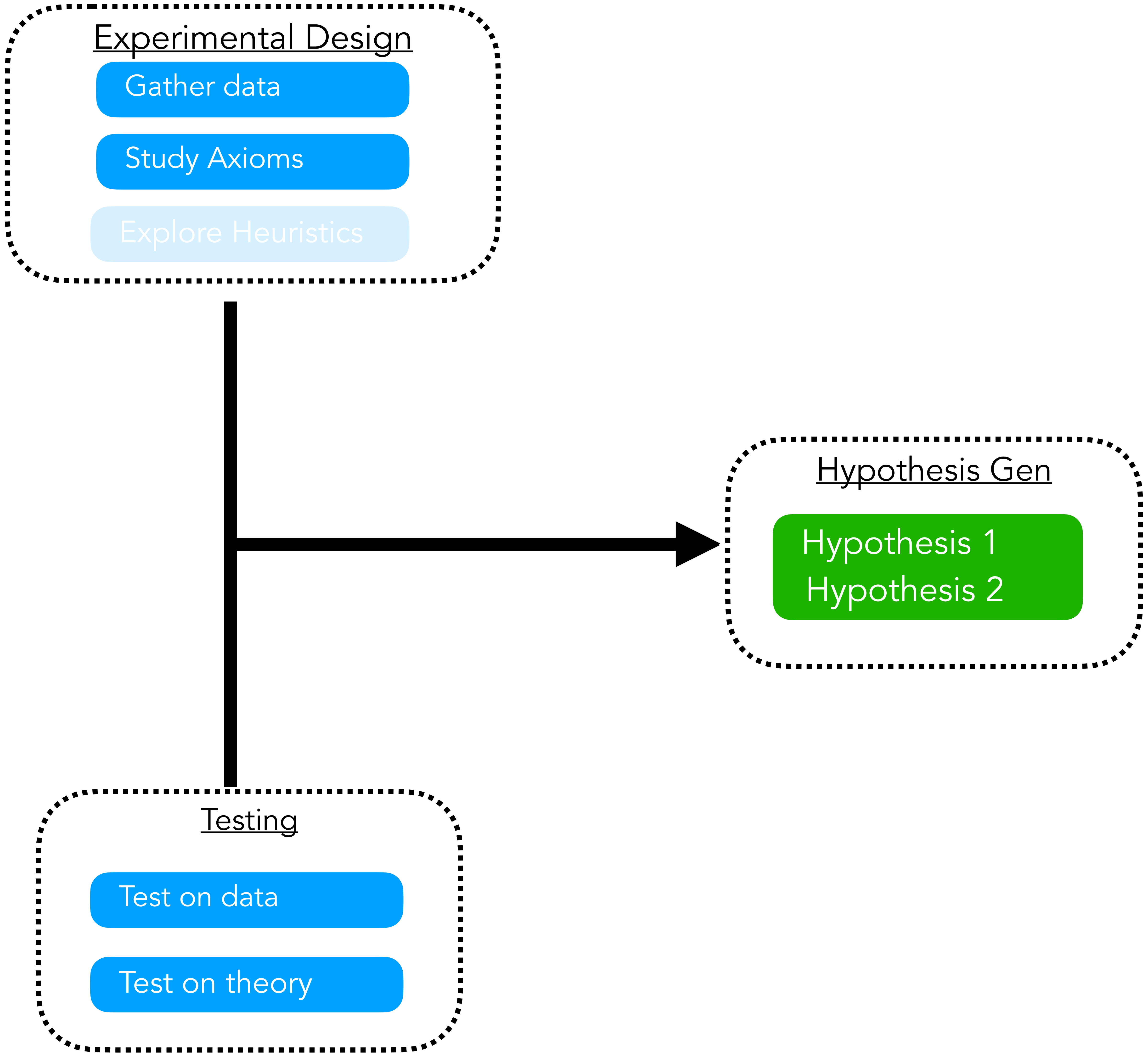
Unified Methods

AI Hilbert [Ryan Cory-Wright, et al. Nature Comms, 2024]

$$\min_{q \in \mathbb{R}_{n,d}[\mathbf{x}]} \sum_{\mathbf{x_i} \in \text{data}} q(\mathbf{x_i}) + \lambda d(q(\mathbf{x}), A)$$

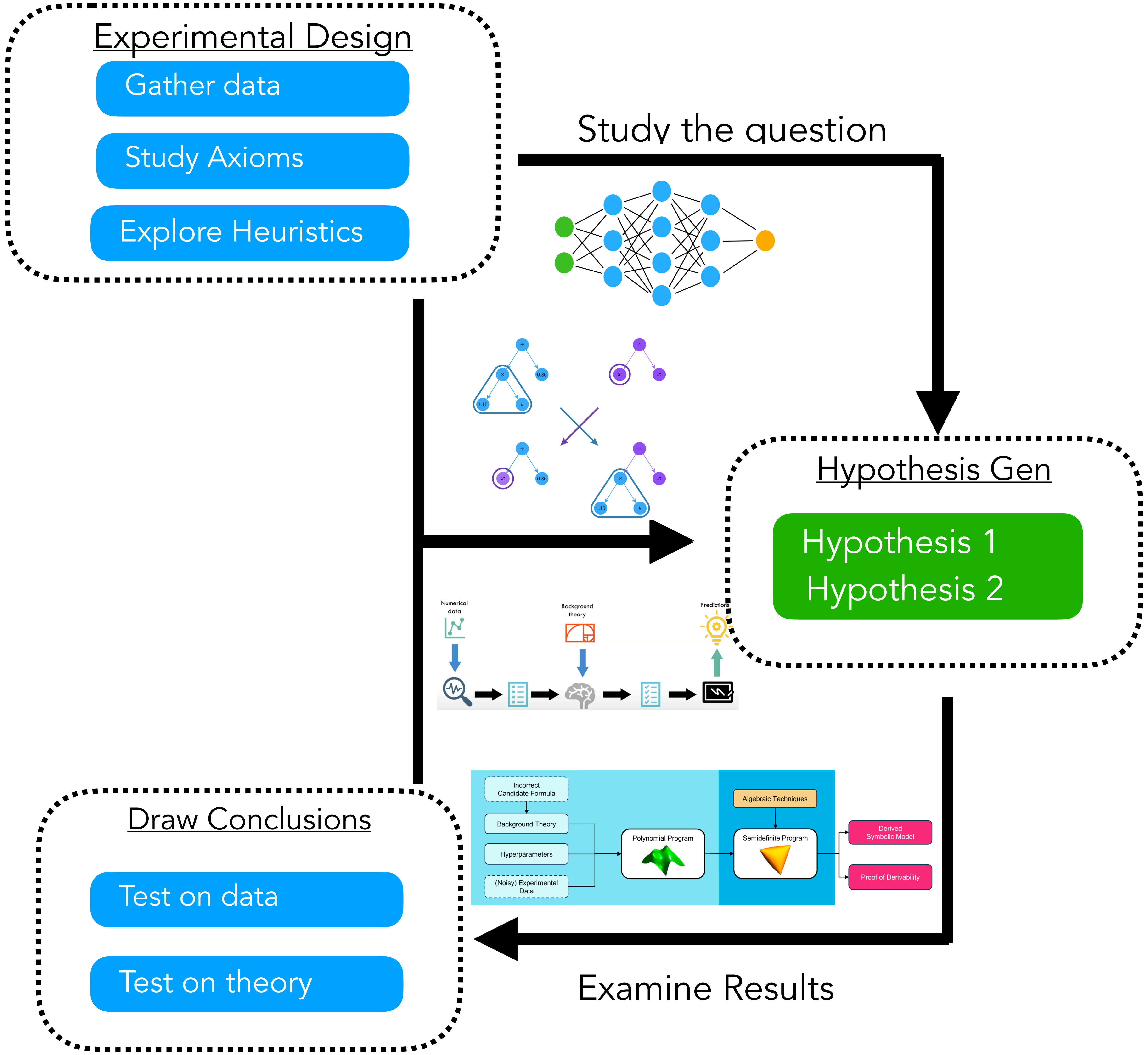
Limitations:

Only works with systems expressible as polynomials. For derivability certificates, we **require axiom systems which contain complete information.**



The Scientific Method

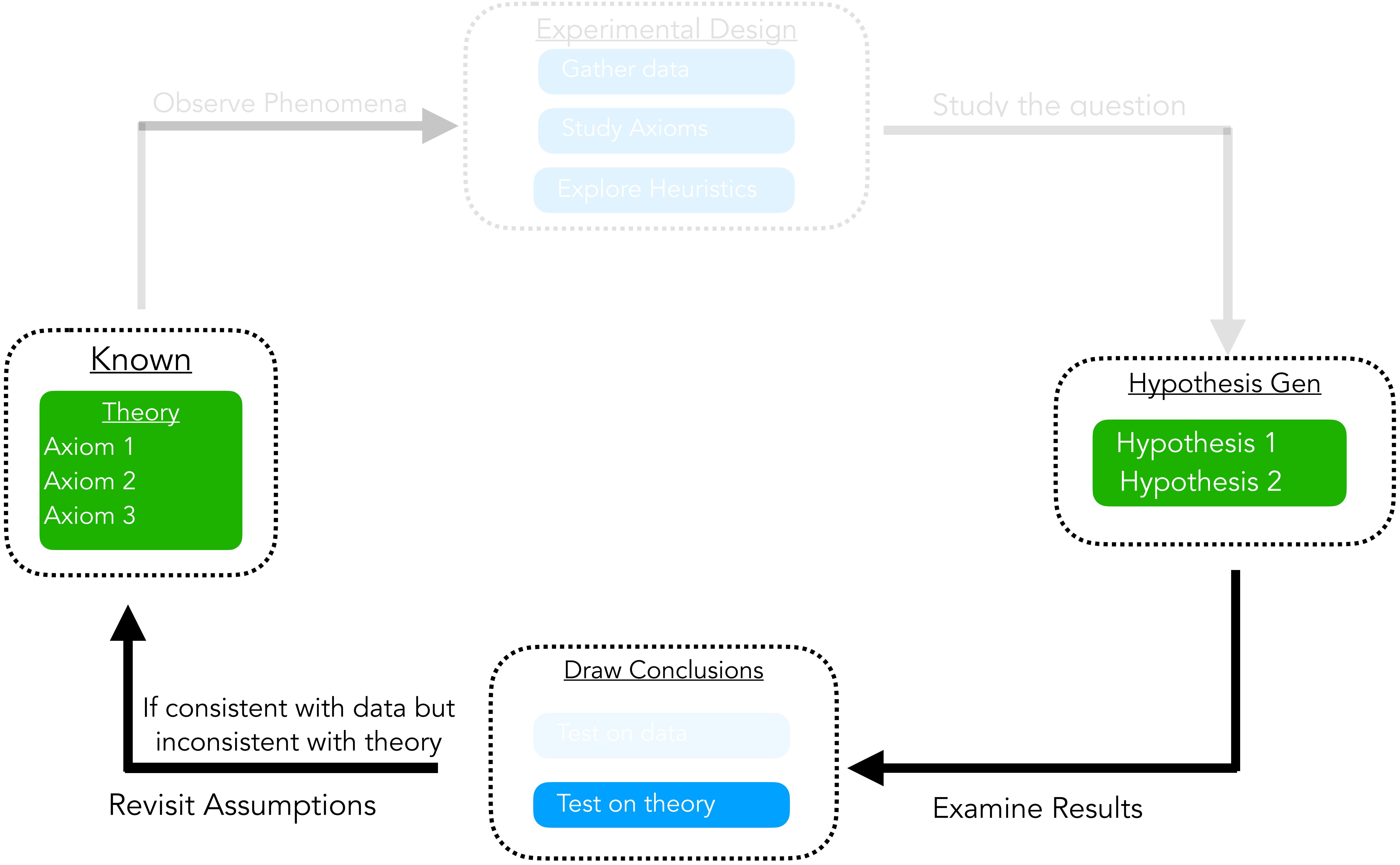
Hypothesis generation methods can either result in expressions for phenomena that are not consistent with theory or are only verifiable if the background theory is complete.



The Scientific Method

Key Question for our work

When theory is inconsistent or incomplete and cannot explain a phenomenon, can we generate corrections in an automated way?

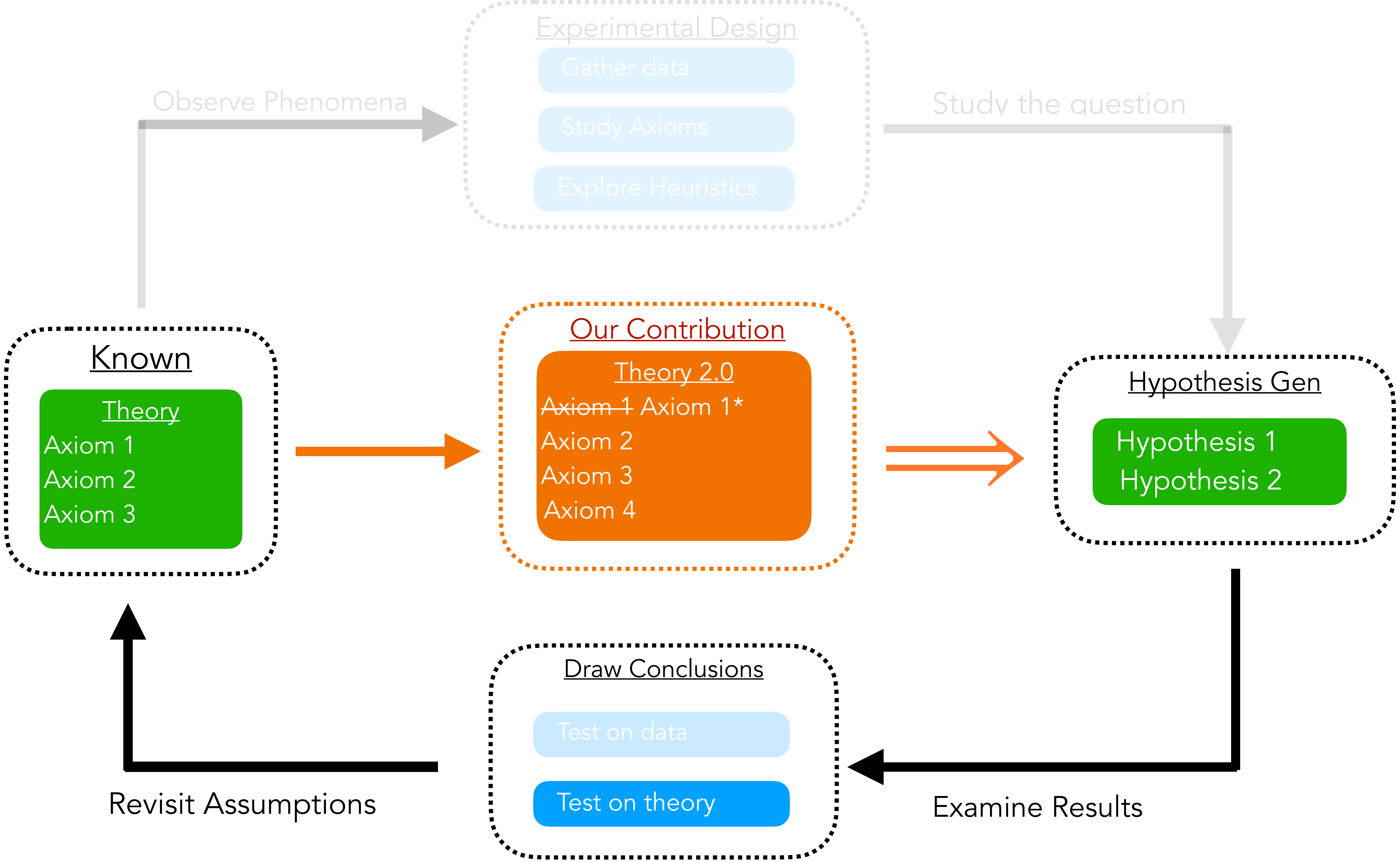


The Scientific Method - Our Contribution

Key Question for our work

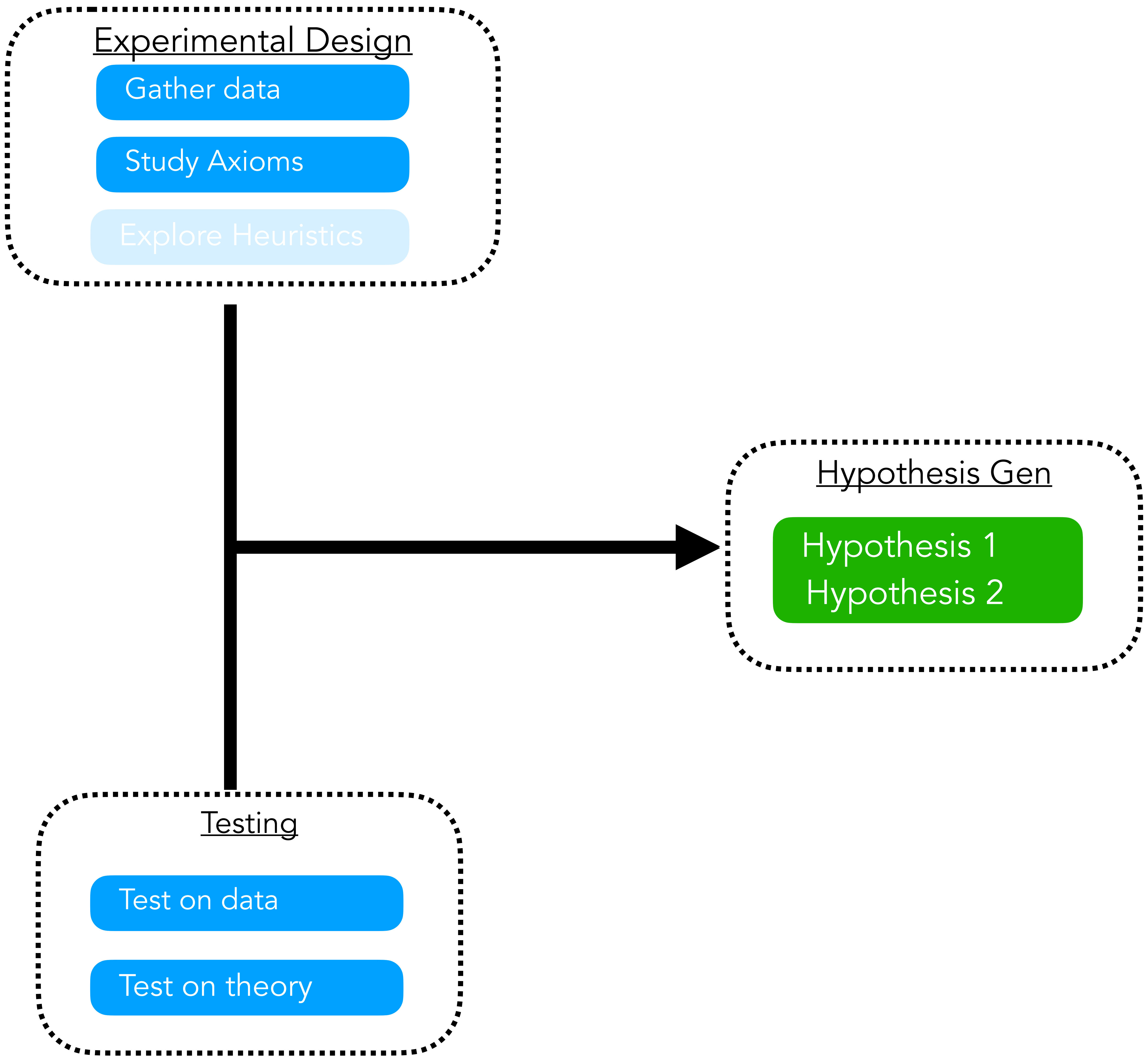
When theory is inconsistent or incomplete and cannot explain a phenomenon, can we generate corrections in an automated way?

Contribution: An automated method of generating candidates for axioms that explain discovered phenomena.



Preliminaries

Rephrasing the discovery problem as a geometric problem.



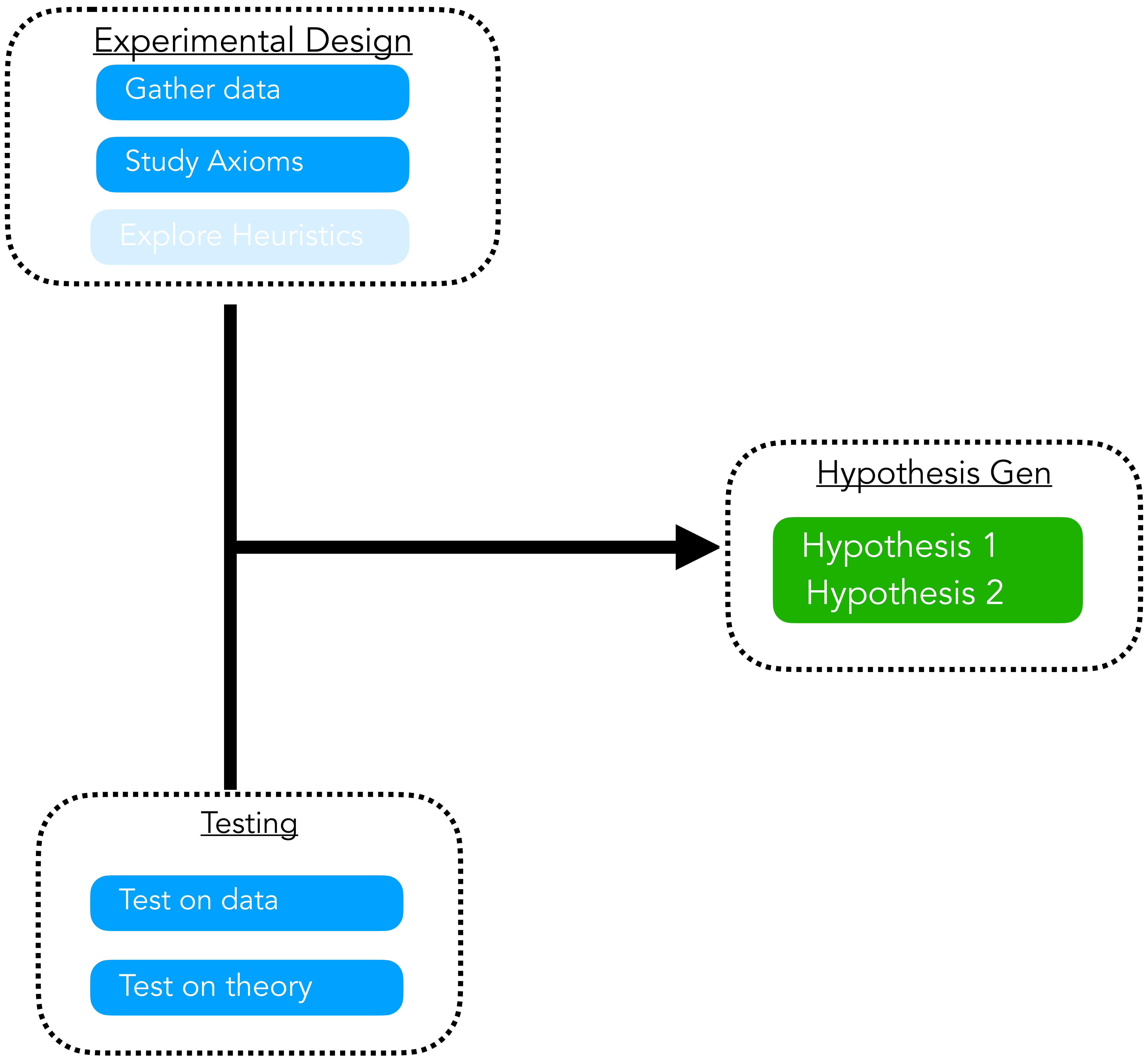
AI Hilbert Review

AI Hilbert

Assumption: all axioms are encoded as polynomials over some basis (traditional indeterminates, trig functions, exponents, etc)

$$\min_{q\in\mathbb{R}_{n,d}[\mathbf{x}]} \sum_{\mathbf{x_i}\in \text{data}} q(\mathbf{x_i}) + \lambda d(q(\mathbf{x}), A)$$

Fit to both theory and background data simultaneously using polynomial optimization.



AI Hilbert Review

Key idea: For polynomials, **we can exactly describe the space of derivable functions.**

Let $A = \{A_1(\mathbf{x}), \ldots, A_k(\mathbf{x})\}$ denote the set of polynomial background axioms.

Putinar's Positive Stellensatz*

A degree $\leq d$ polynomial $q(\mathbf{x})$ vanishes on the solution set $\mathscr{H} = \{\mathbf{x} \in \mathbb{R}^n : A_i(\mathbf{x}) = 0 \text{ for each } i\}$ if and only if for some degree $\leq d$ polynomials $\alpha_1(\mathbf{x}), \ldots, \alpha_k(\mathbf{x})$, we have

$$q(\mathbf{x}) = \sum_{i=1}^{k} \alpha_i(\mathbf{x}) A_i(\mathbf{x})$$

Takeaway: **we can express algebraic derivability from axioms as algebraic combinations of axioms**!

*Putinar's positive stellensatz is actually a very similar statement about semi algebraic sets allowing for inequalities. We cite Putinar for continuity with AI Hilbert, but we will use this more restricted version without inequalities. You can also see: Hilbert's Nullstellensatz, which is the generalization of this statement for arbitrary degrees.

AI Hilbert Review

Example: Kepler's Third Law of Planetary Motion

Axioms

$$d_1m_1 - d_2m_2 = 0$$
$$(d_1 + d_2)^2F_g - Gm_1m_2 = 0$$
$$F_c - m_2d_2w^2 = 0$$
$$F_c - F_g = 0$$
$$wp - 1 = 0$$

$$\implies$$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{G(m_1 + m_2)}}$$

or

$$m_1m_2Gp^2 - m_1d_1d_2^2 - m_2d_1^2d_2 - 2m_2d_1d_2^2 = 0$$

Instead of writing a traditional derivation of Kepler from the axioms, we could write it as the following combination

$$-d_2^2p^2w^2$$
$$-p^2$$
$$d_1^2p^2 + 2d_1d_2p^2 + d_2p^2$$
$$d_1^2p^2 + 2d_1d_2p^2 + d_2p^2$$
$$(pwd_1d_2 + d_1d_2)(m_1d_2 + m_2d_1 + 2m_2d_2)$$

AI Hilbert Review

Define the distance between a potential candidate hypothesis

$$d(q(\mathbf{x}), A) = \min_{\alpha_i \in \mathbb{R}_{n,d}[\mathbf{x}]} \text{coeff}\|q(\mathbf{x}) - \sum_{i=1}^{k} \alpha_i(\mathbf{x}) A_i(\mathbf{x})\|$$

So we can solve the optimization problem

$$\min_{q \in \mathbb{R}_{n,d}[\mathbf{x}]} \sum_{\mathbf{x_i} \in \text{data}} q(\mathbf{x_i}) + \lambda d(q(\mathbf{x}), A)$$

This can be computationally expensive (e.g. Radiational Gravitational Wave Equation took 640GB memory, >5 hours on MIT SuperCloud). First motivation to bring in geometry: can we speed this up?

Converting Discovery to Geometry

The discovery problem with complete theory is a projection problem.

Axioms

$$d_1m_1 - d_2m_2 = 0$$
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$$F_c - m_2d_2w^2 = 0$$
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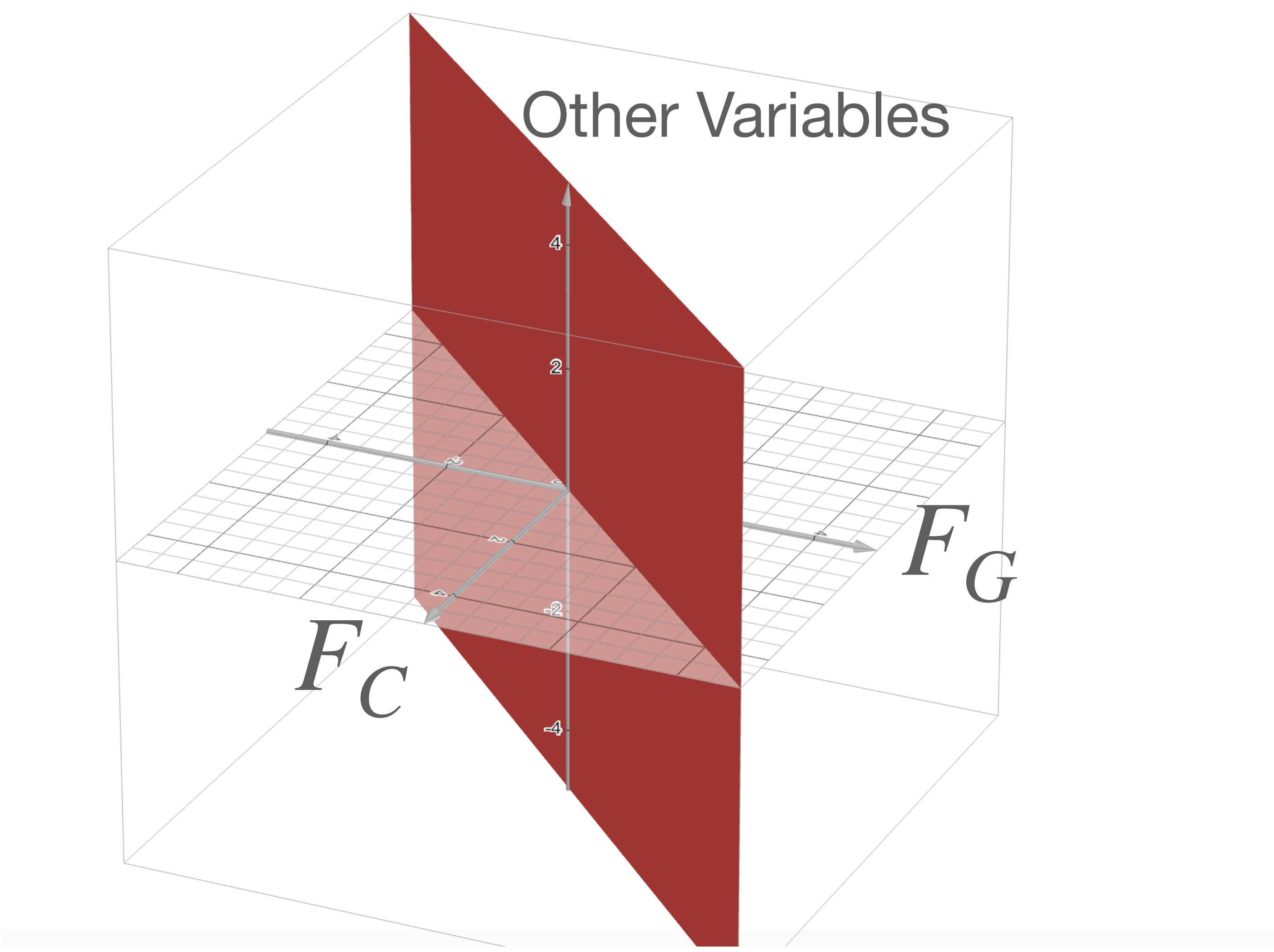
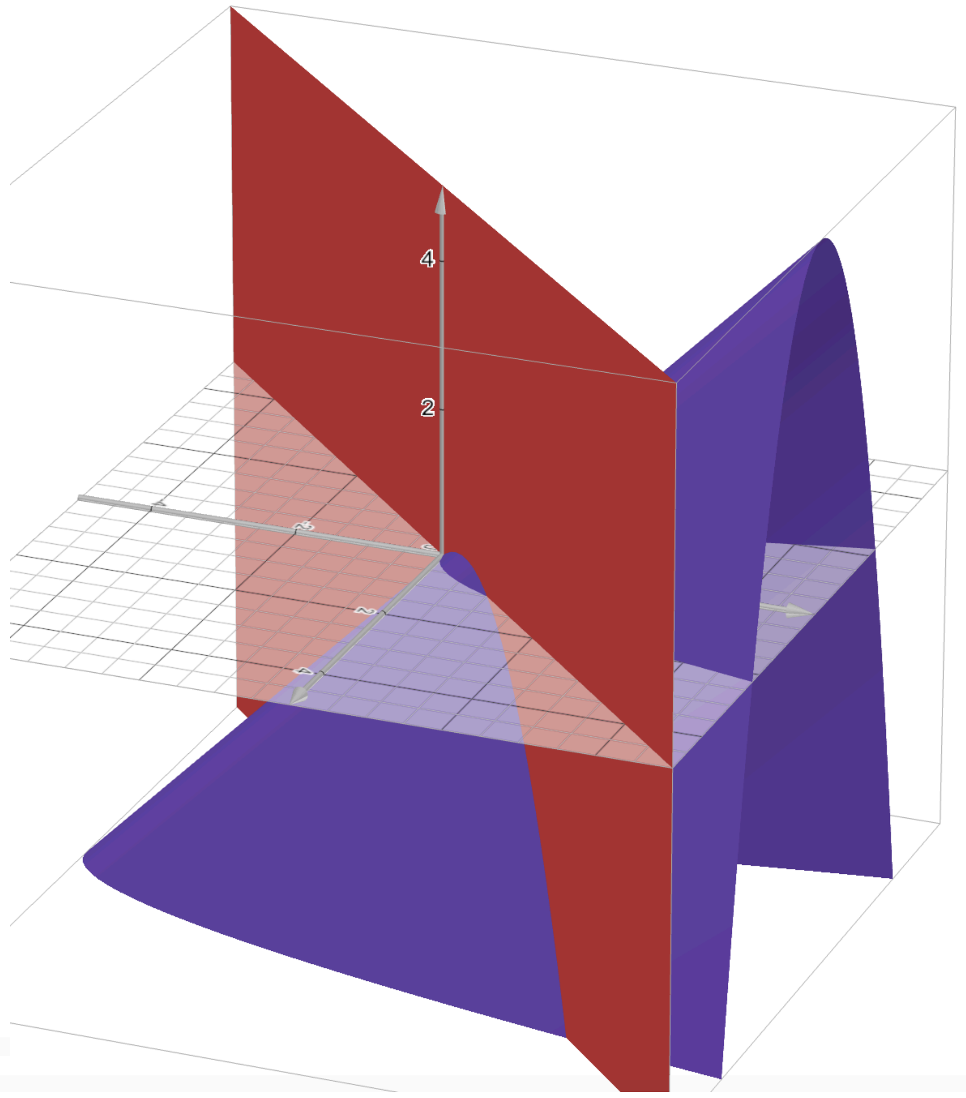
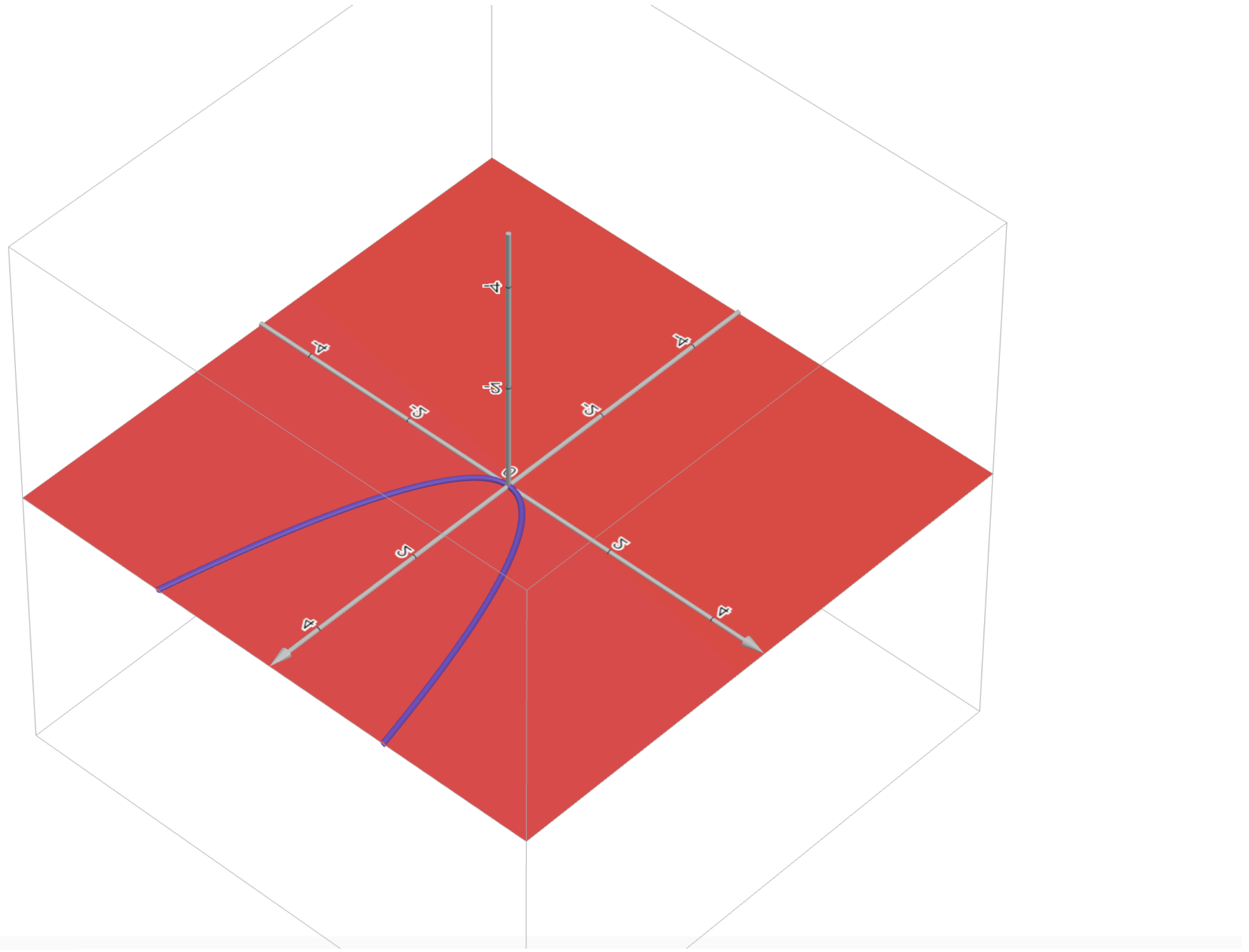
$\Longrightarrow$

Phenomenon

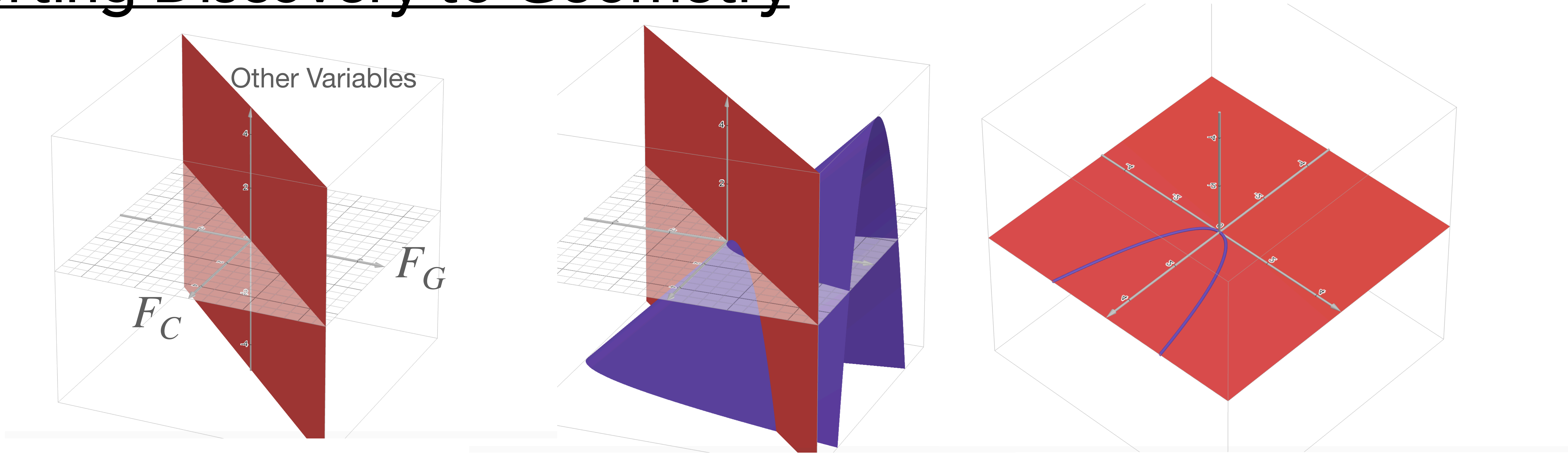
$$p = \sqrt{\frac{4(d_1 + d_2)^3}{G(m_1 + m_2)}}$$







Converting Discovery to Geometry



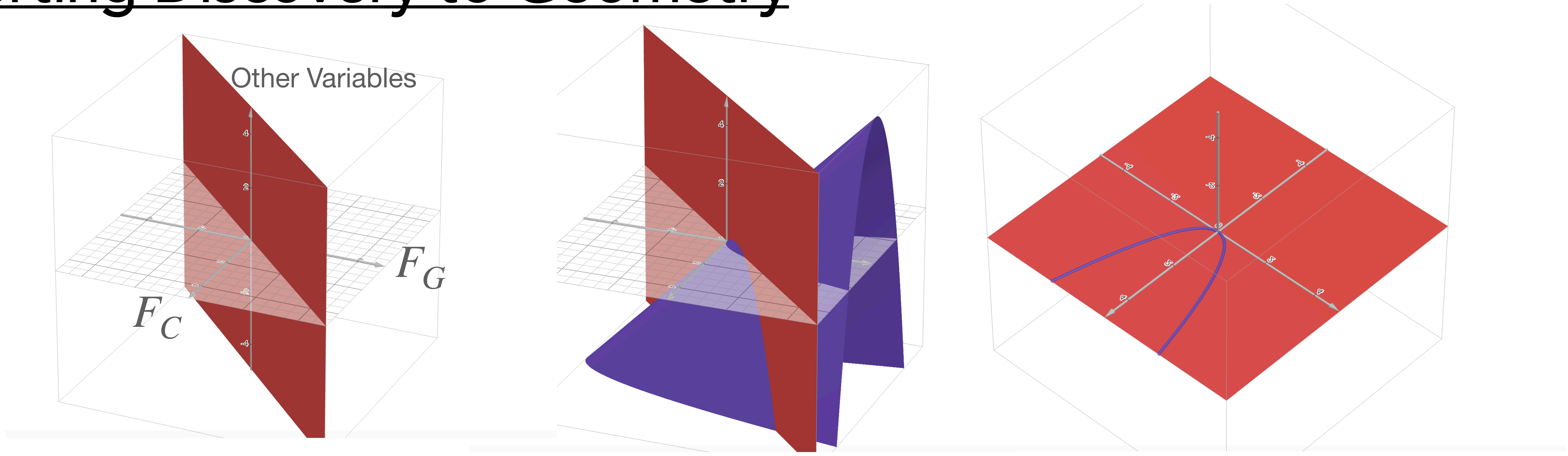
Variety

The set of solutions to a system of polynomial equations $A_1, \ldots, A_k$ is called an algebraic variety $V(A_1, \ldots, A_k)$

What this captures

The solution set for which we will take projections to be the new search space

Converting Discovery to Geometry



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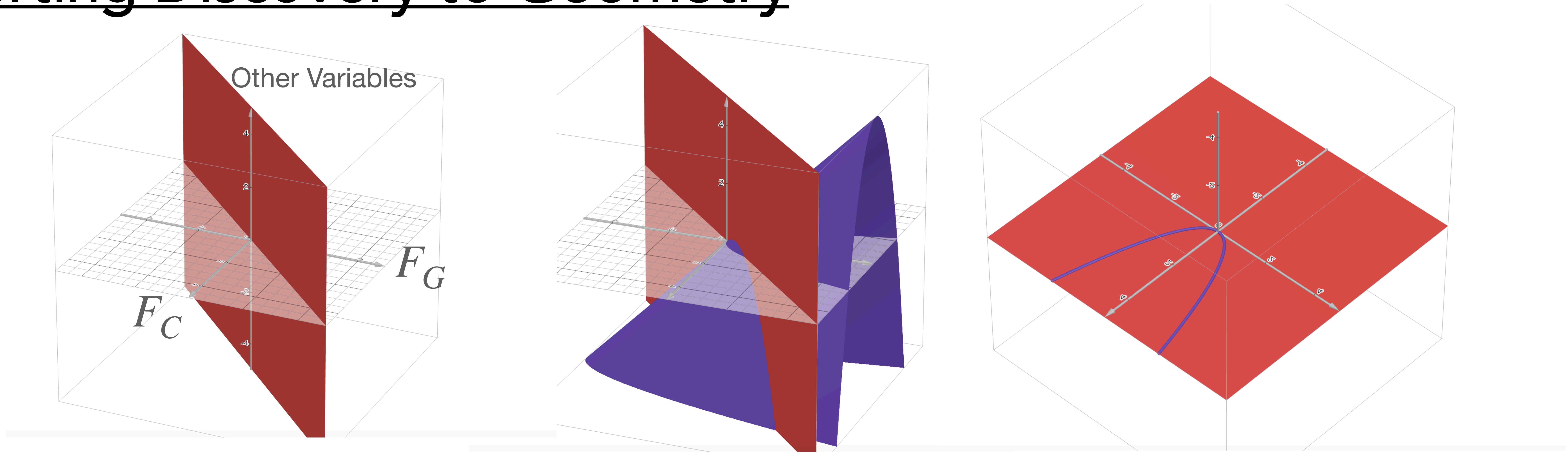
Stellensatz

Every polynomial q that vanishes over the solution set of a system of polynomials $A_1, \ldots, A_k$ is an algebraic combination.

Converting Discovery to Geometry



Variety

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What this captures

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Ideals

The ideal generated by a polynomial system $A_1, ..., A_k$ is the set of all algebraic combinations. It is denoted $\langle A_1, ..., A_k \rangle$.

What this captures

A closed form expression of the space of all phenomena that are algebraic consequences of the axioms.

Converting Discovery to Geometry

How does this help with projection? The following facts and theorems help make this work

Fact 1: An algebraic consequence of axioms will not generate new solution points. I.e. $V(A_1, \ldots, A_k) = V(\langle A_1, \ldots, A_k \rangle)$

$$(d_1 + d_2)^2 F_g - G m_1 m_2 = 0$$
$$F_c - m_2 d_2 w^2 = 0$$
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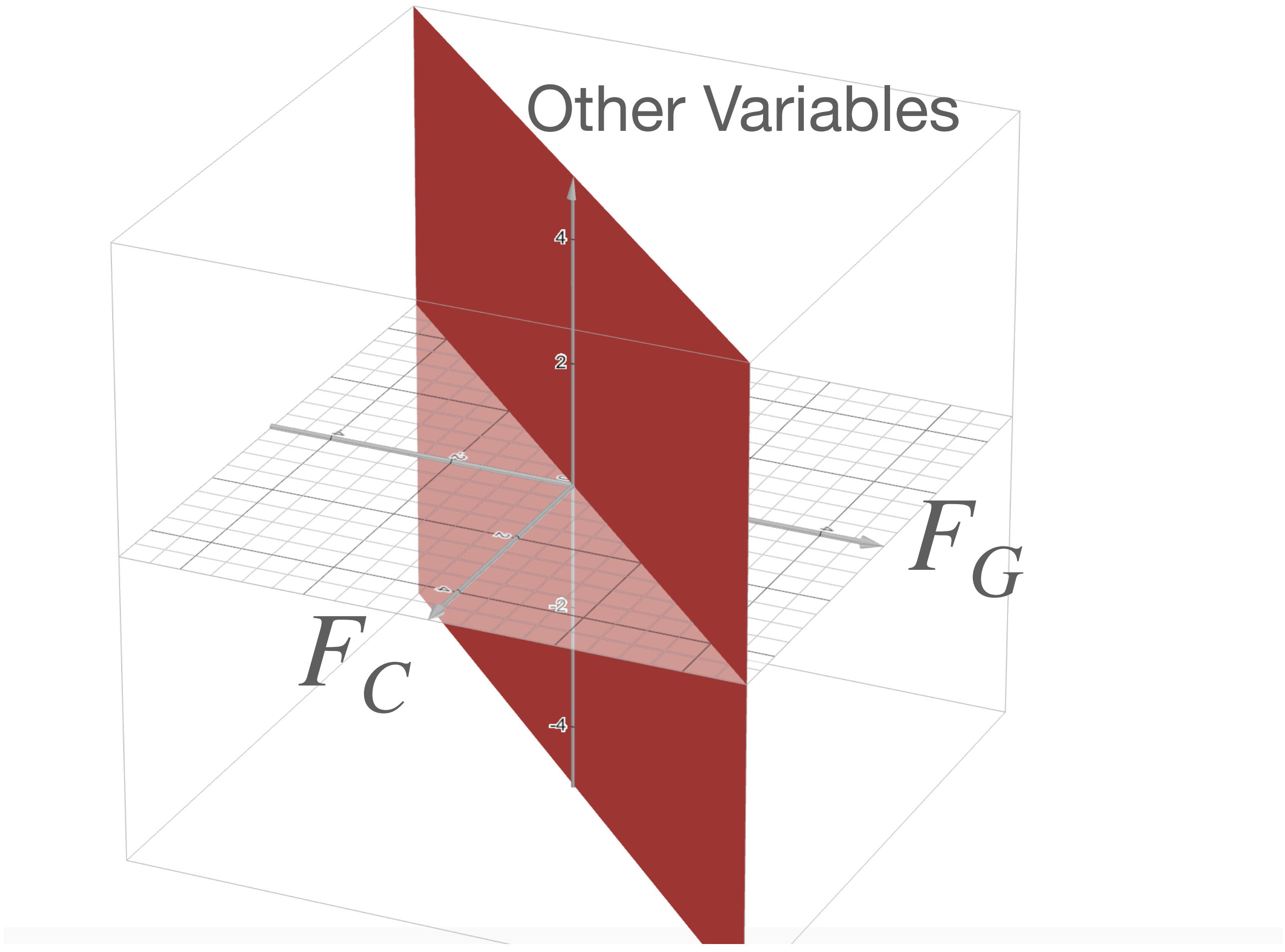
$$wp - 1 = 0$$



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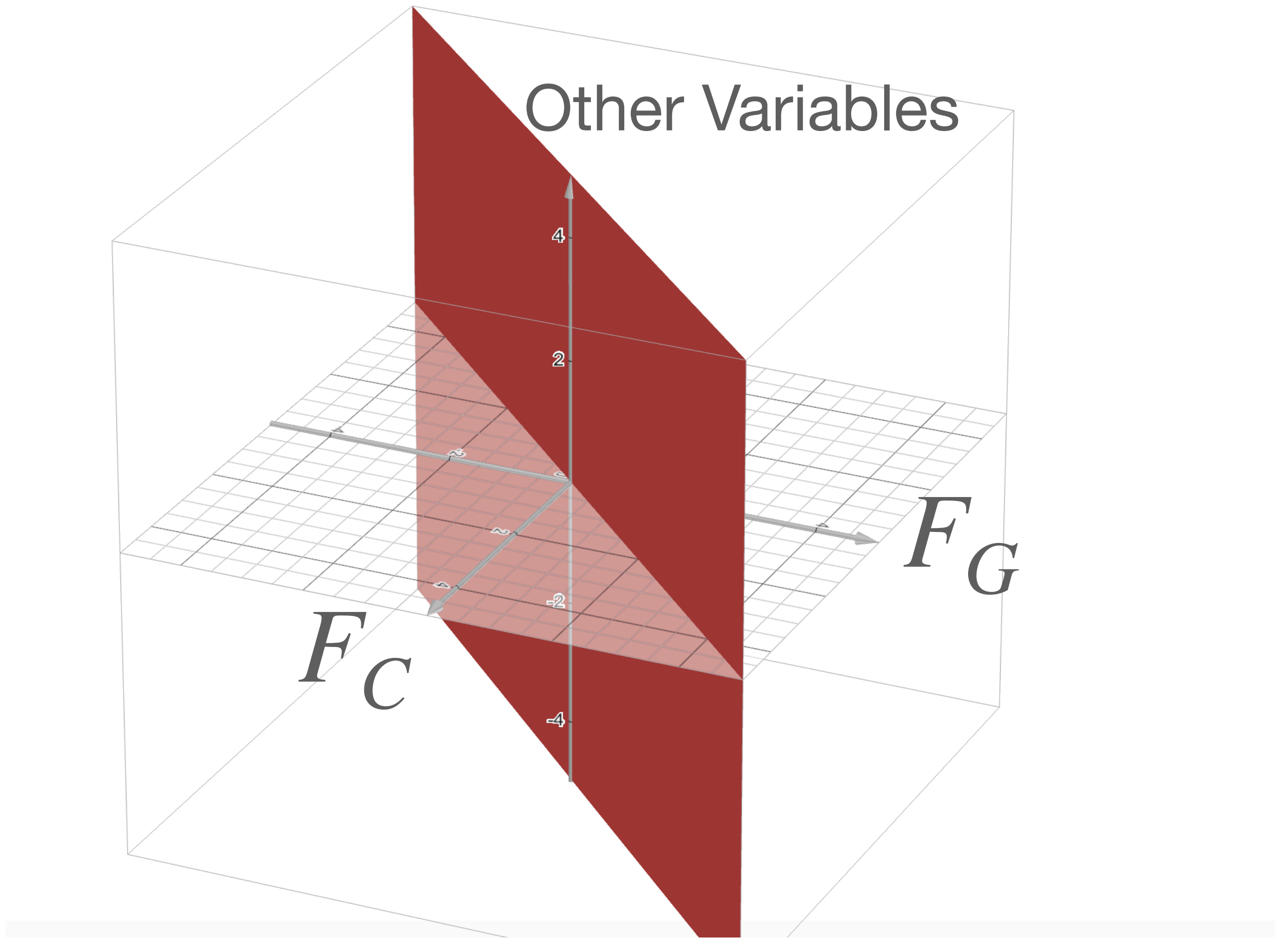

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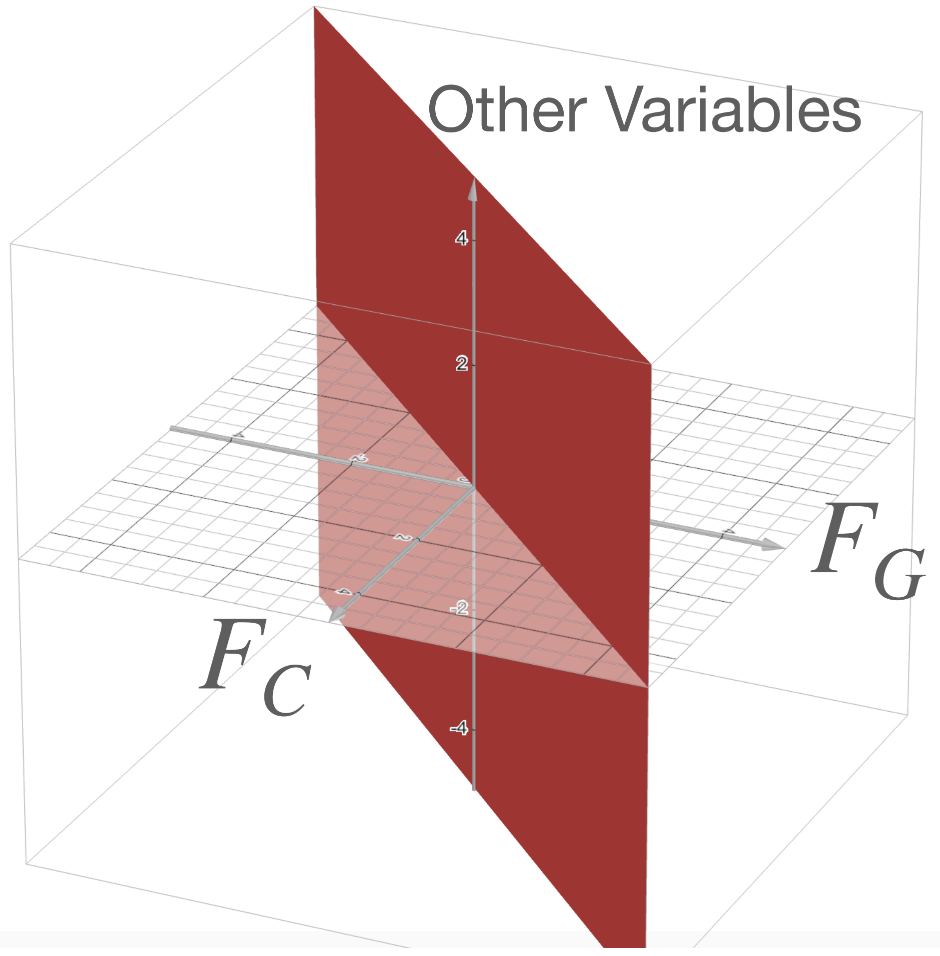
Fact 2: There can be multiple generating sets for the same ideal. The generators are not unique.

$$\begin{gathered} (d_1 + d_2)^2 F_g - Gm_1m_2 = 0 \\ F_c - m_2d_2w^2 = 0 \\ F_c - F_g = 0 \\ wp - 1 = 0 \end{gathered}$$



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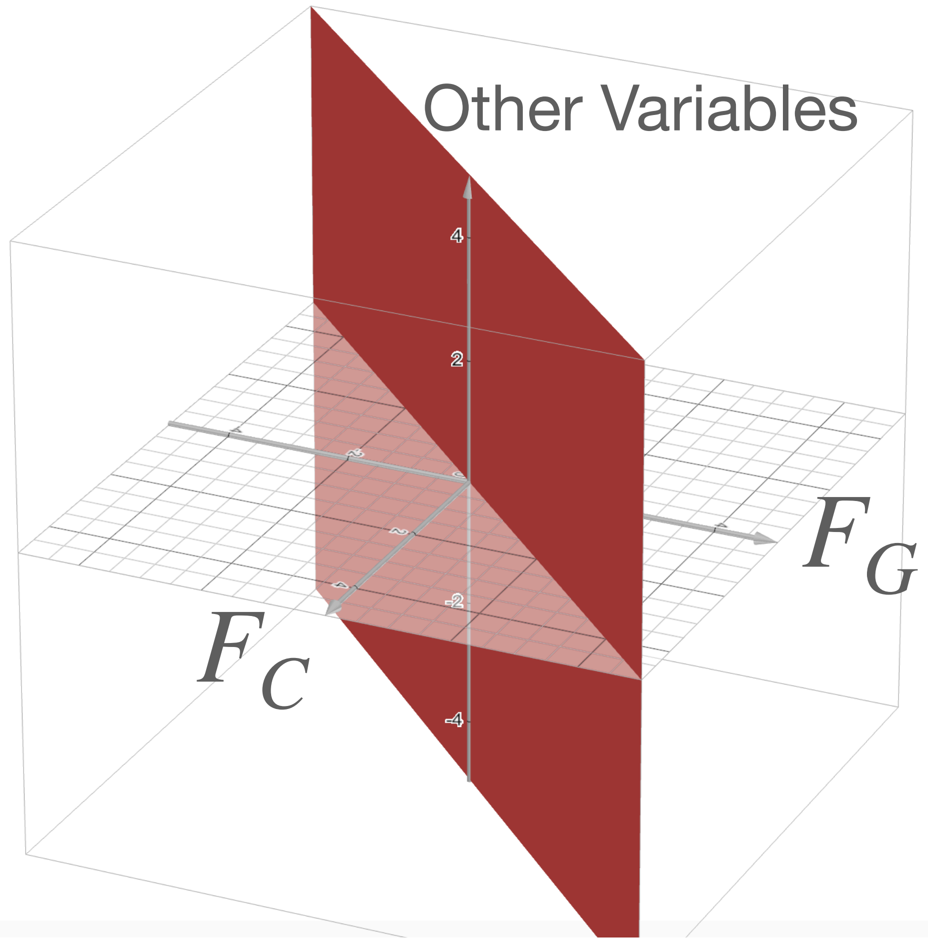

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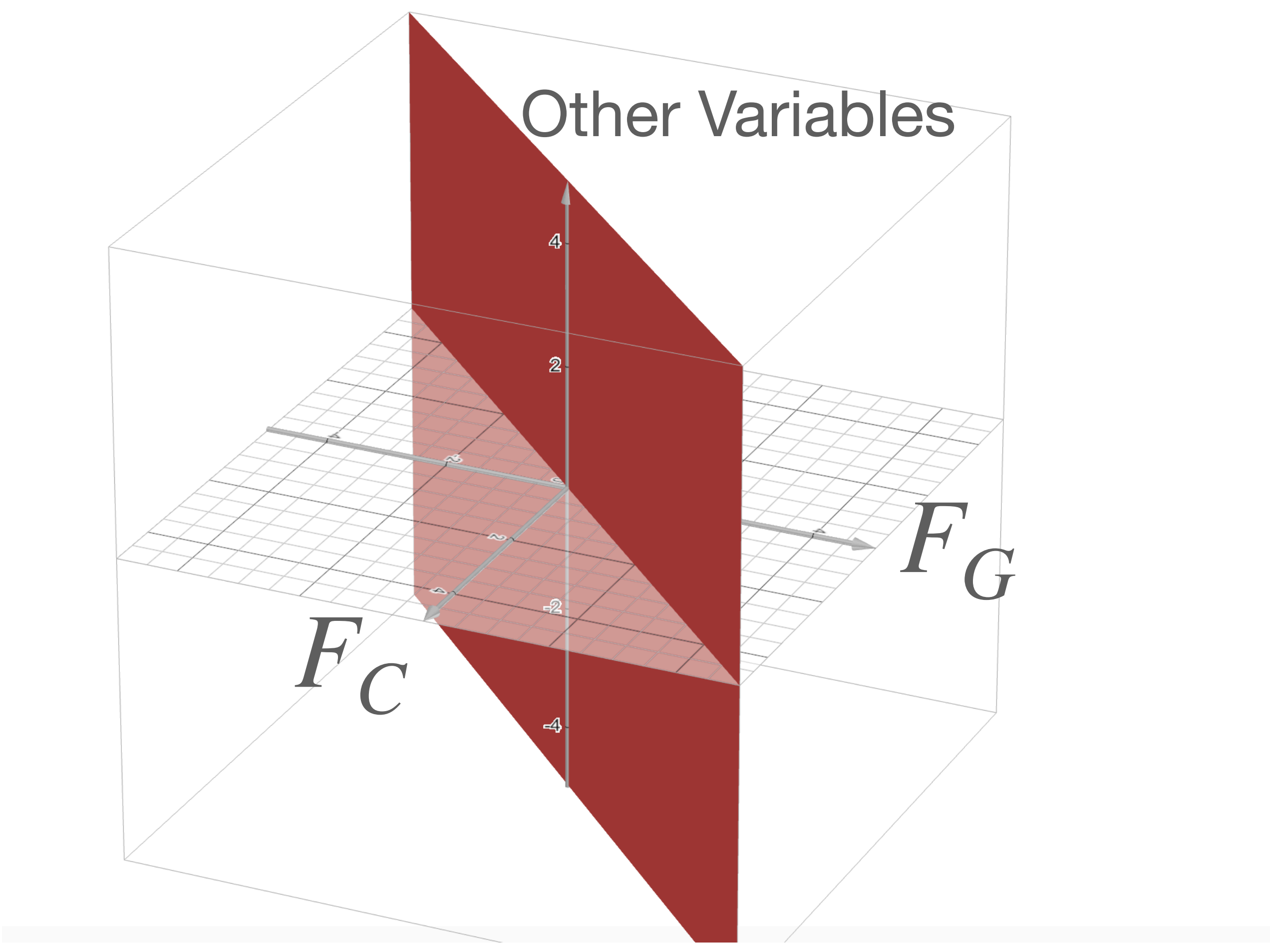
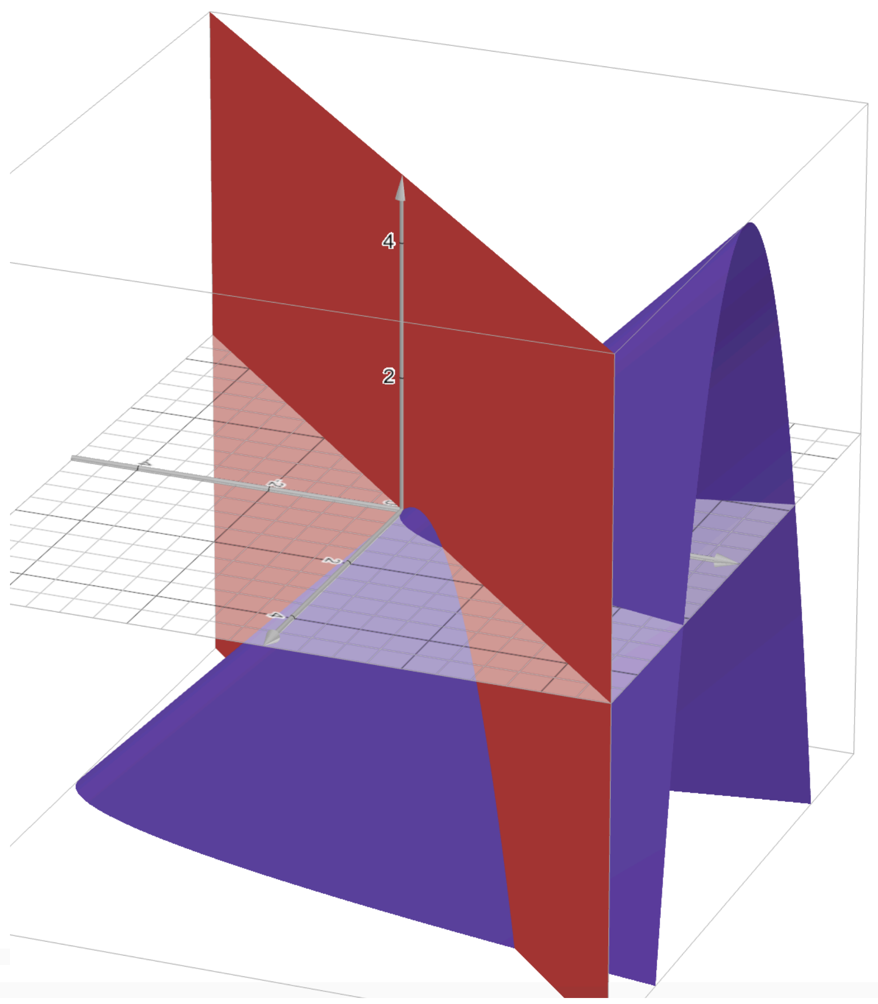
Fact 2: There can be multiple generating sets for the same ideal. The generators are not unique.

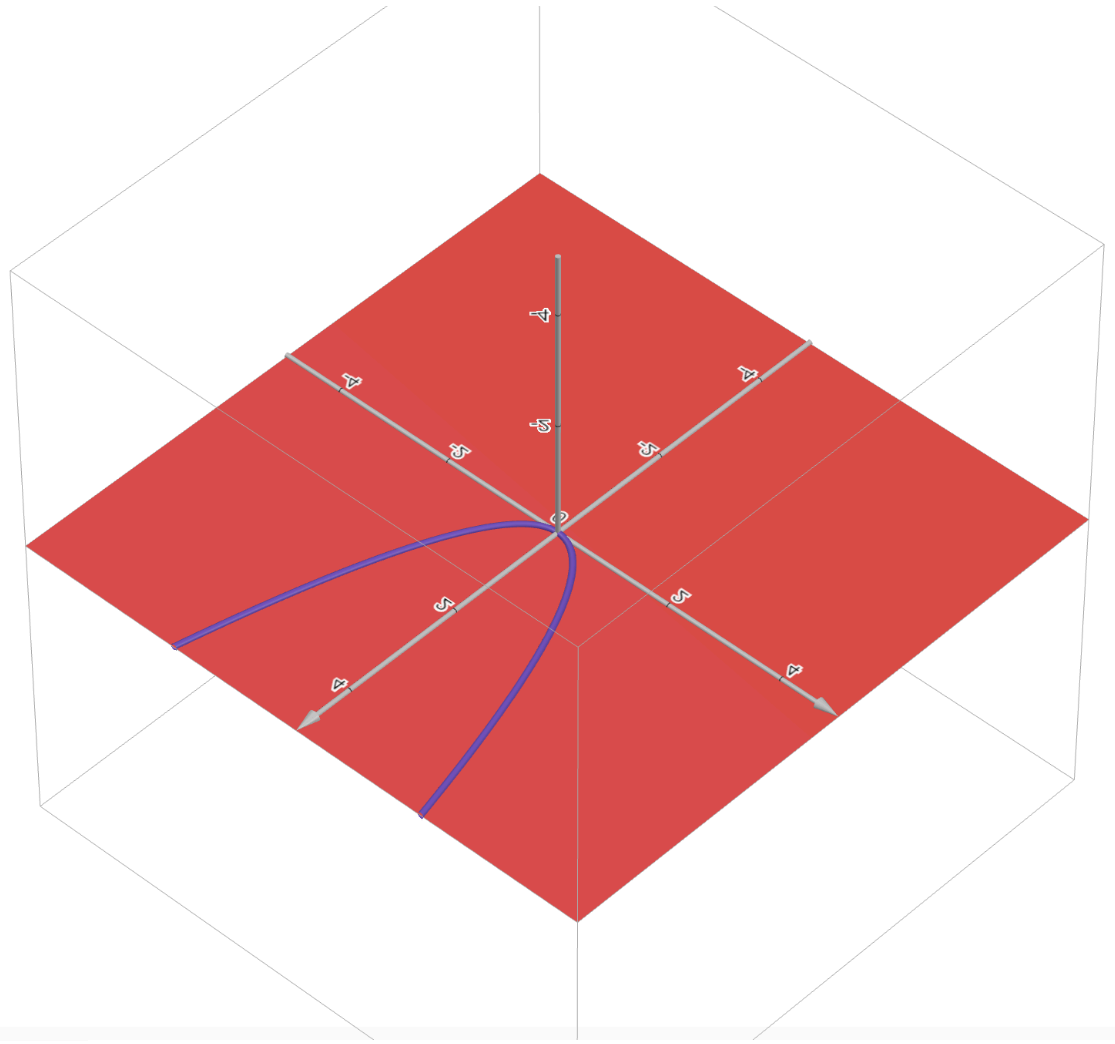
Elimination Theorem

There exists a unique generating set, called a Gröbner Basis, $\mathscr{G} = \{g_1, ..., g_M\}$ of $\langle A_1, ..., A_k \rangle$ such that

$$\langle A_1, ..., A_k \rangle \cap \mathbb{R}[x_1, ..., x_r] = \{g_1, ..., g_M\} \cap \mathbb{R}[x_1, ..., x_r]$$

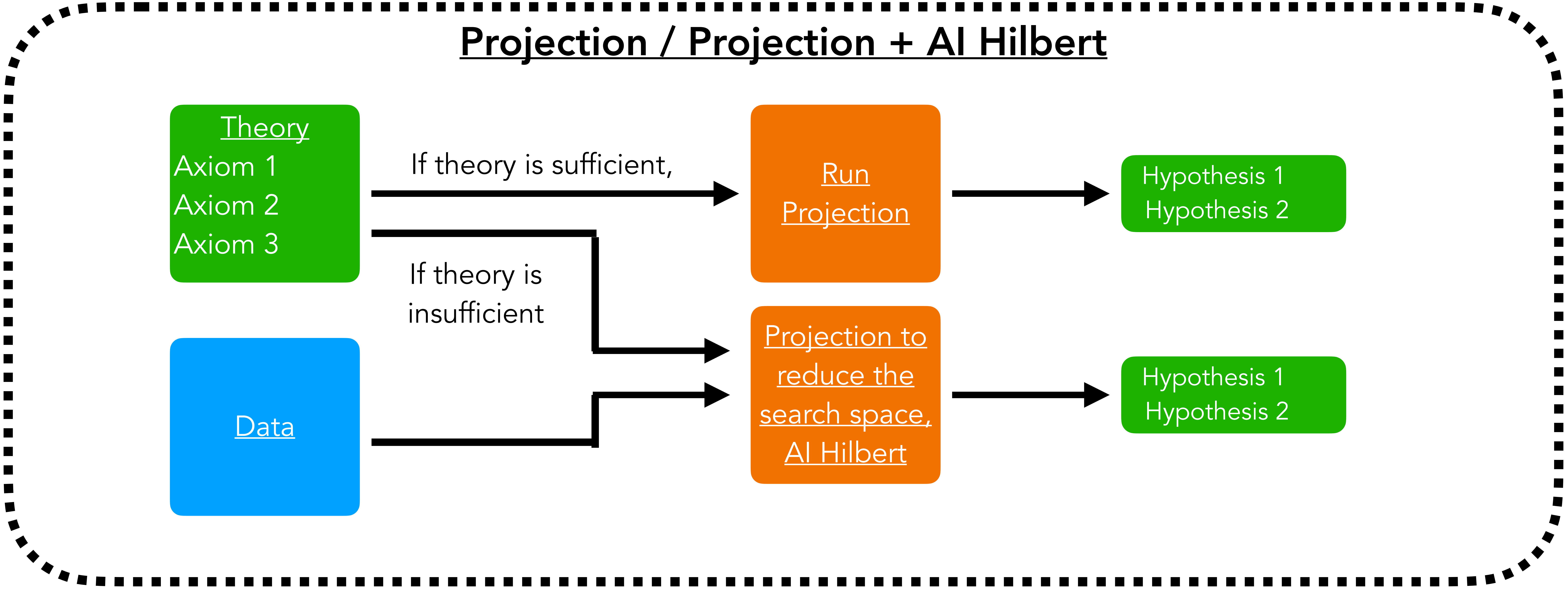






Converting Discovery to Geometry

This gives us the following options:

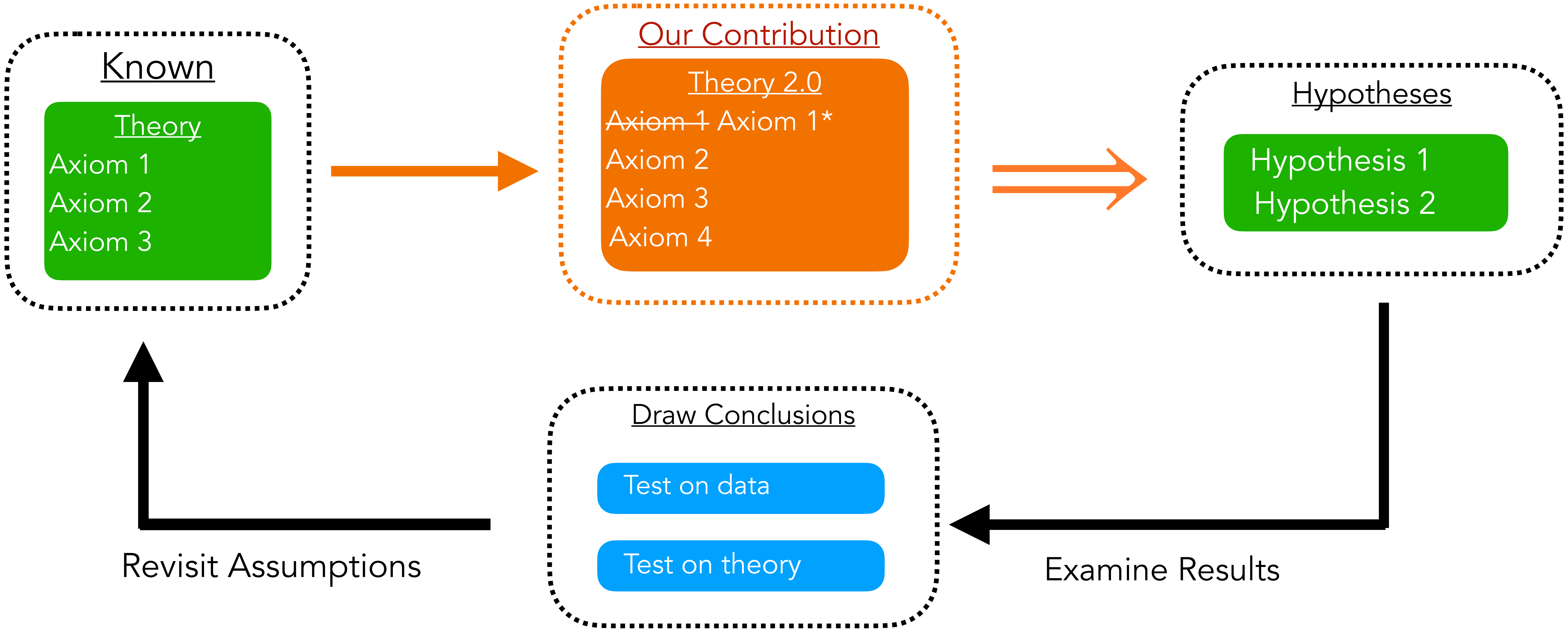


Converting Discovery to Geometry - Results

What this does for us: we can achieve a reduction in LP size and therefore speedup for AI Hilbert.

Problem	AI Hilbert			Projection		Projection+Hilbert		
	Found	Monoms	Time	Found	Basis Size	Found	Monoms	Time
Grav Waves	✓	487000	14656s*	✓	1	✓	12778	32.7s
Kepler	✓	4048	5.12s	✓	3	✓	435	0.04s
Kepler w/o axiom 1	✓	4048	5.23s	✓	1	✓	435	0.05s
Compton Scattering	✓	54264	1789s	✓	1	✓	924	1.9s
Light Damping	✓	58170	125s	✓	1	✓	1439	0.8s
Light Damping**	?	8450393	T/o CPU^{+}	✓	1	✓	1439	0.8s
Neutrino Decay	✓	2643	3.4s	✓	1	✓	54	0.2s
Escape Velocity	✓	5832	4.3s	✓	1	✓	210	0.1s
Escape Velocity***	?	774753	T/o CPU^{+}	✓	1	✓	210	0.1s
Hall Effect	✓	1045830	T/o CPU^{++}	✓	1	✓	424	11.1s
Inelastic Collision	✓	41754	123s	✓	1	✓	81	0.3s
Hagen Poiseuille	✓	4140	4.2s	✓	1	✓	756	0.5s
Einstein	✓	5634	1.5s	✓	1	✓	81	0.1s

Abductively Inferring Axioms from an Incomplete or Incorrect System.



Abductive Inference

$$
\begin{gathered}
(d_1 + d_2)^2 F_g - m_1 m_2 = 0 \\
F_c - m_2 d_2 w^2 = 0 \\
F_c - F_g = 0 \\
wp - 1 = 0
\end{gathered}
\implies
p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}
$$

Abductive Inference

~~$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$~~

$$F_c - m_2 d_2 w^2 = 0$$

$$F_c - F_g = 0$$

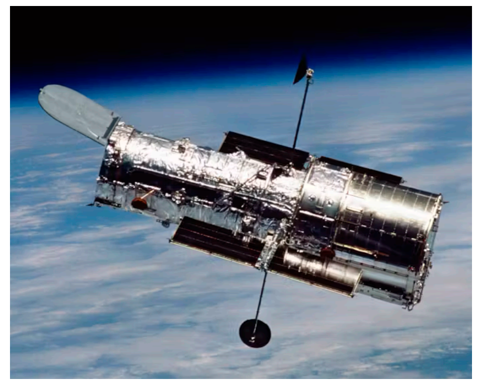
$$wp - 1 = 0$$

$$\not\Longrightarrow$$

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

Abductive Inference

Data



+

~~$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$~~

$$F_c - m_2 d_2 w^2 = 0$$

$$F_c - F_g = 0$$

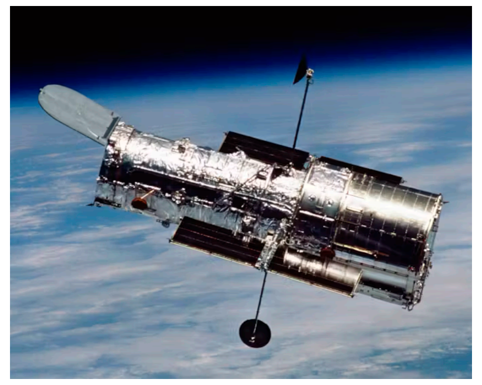
$$wp - 1 = 0$$

→

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

Abductive Inference

Data



+

~~$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$~~

$F_c - m_2 d_2 w^2 = 0$

$F_c - F_g = 0$

$wp - 1 = 0$

→

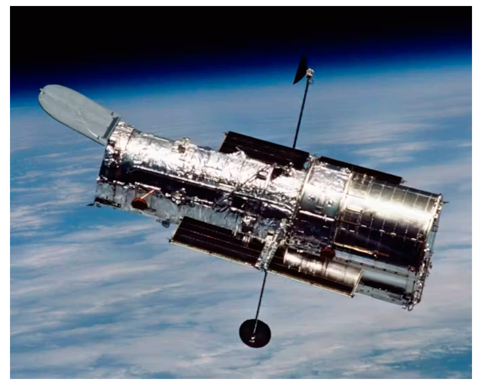
$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

→

$$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$$

Abductive Inference

Data



+

~~$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$~~

$F_c - m_2 d_2 w^2 = 0$

$F_c - F_g = 0$

$wp - 1 = 0$

→

$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$

→

$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$

Aim: Given a polynomial phenomenon Q and a polynomial axiom system $A_1, \ldots, A_{k-1}$ which does not derive Q, generate a list of candidate polynomials $\{\hat{A}_k^i\}$ such that $A_1, \ldots, A_{k-1}, \hat{A}_k^i$ do prove Q for each i.

Constraints:

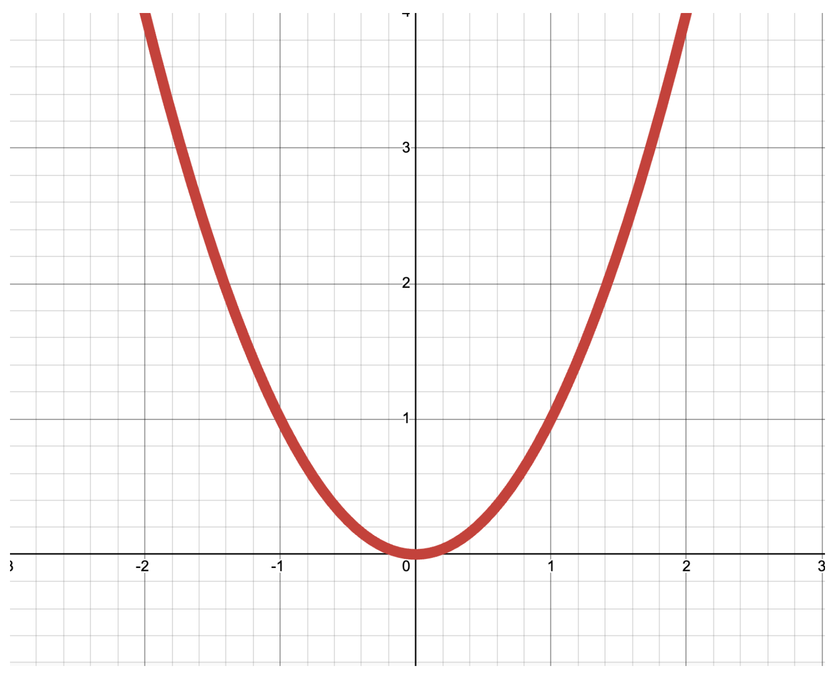
1. We assume we do not have any data for A_k (otherwise this would be equivalent to hypothesis generation and we could fit to data)
2. We assume no knowledge about the exact variables over which A_k is defined other than it is defined over a subset of variables of the entire system (otherwise we could either gather data or project varieties as before)
3. We assume that A_k is "simpler" than Q - we will make this more precise in a few slides. This is to avoid the trivial case $A_k = Q$.
4. We will generalize to multiple missing axioms, but for now still stick to one for illustration.

Abductive Inference

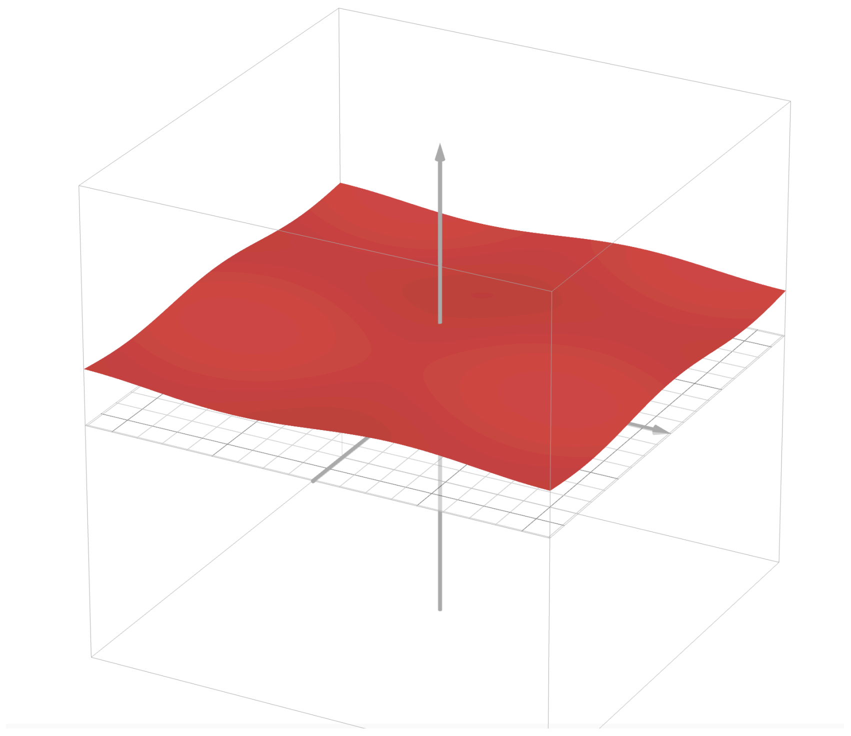
Key idea: Reducibility introduced by Q to the known axioms tells us about residuals.

Irreducible Varieties

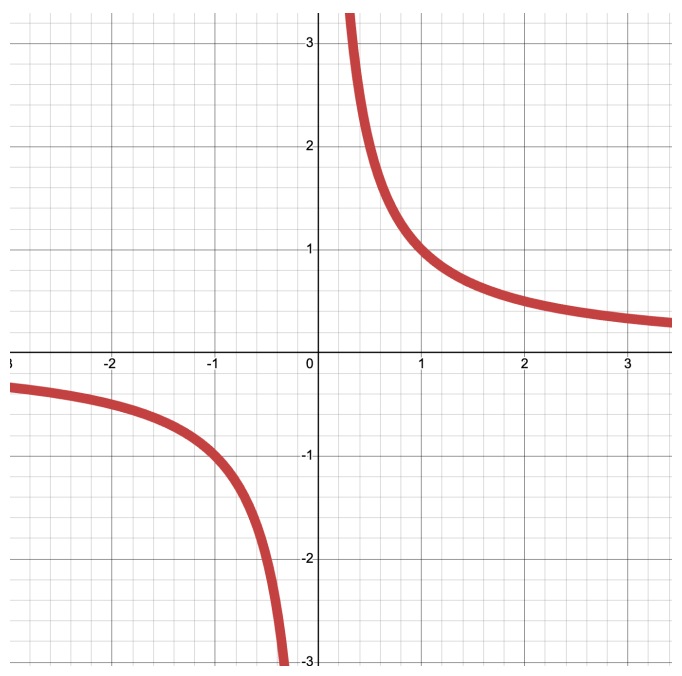
A variety V is irreducible if it cannot be written as the union of two smaller varieties.



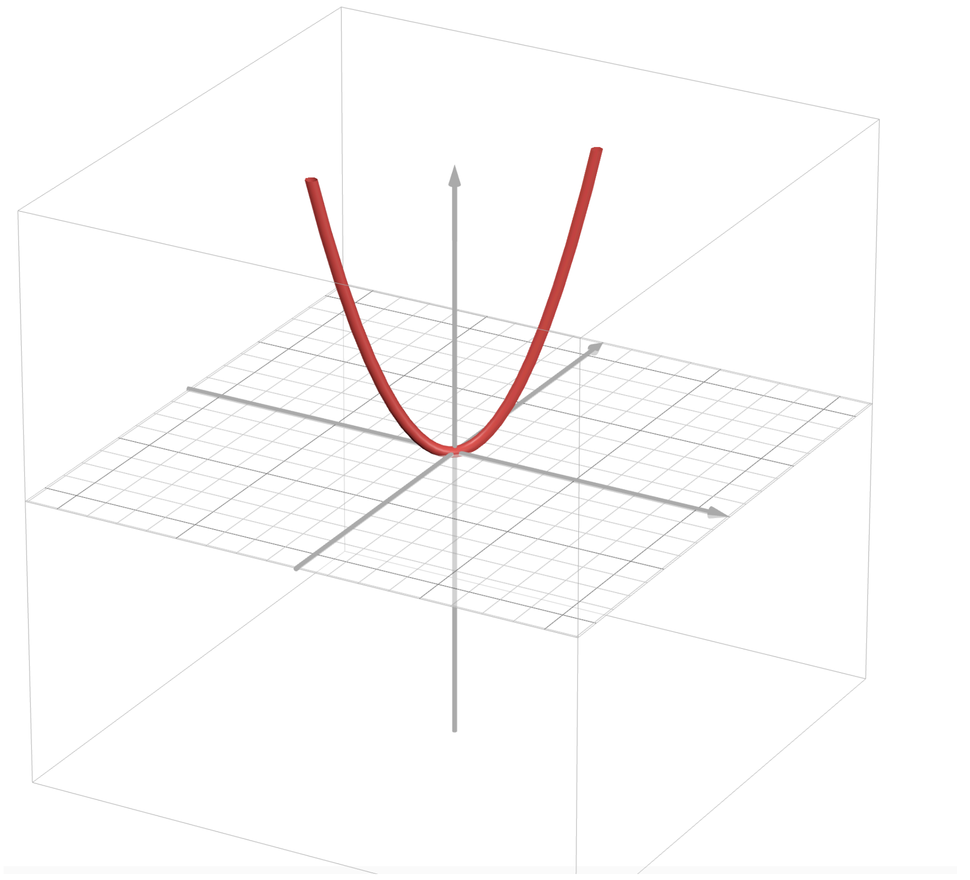
$V(y - x^2)$



$V(x + y + z - 1)$



$V(xy - 1)$



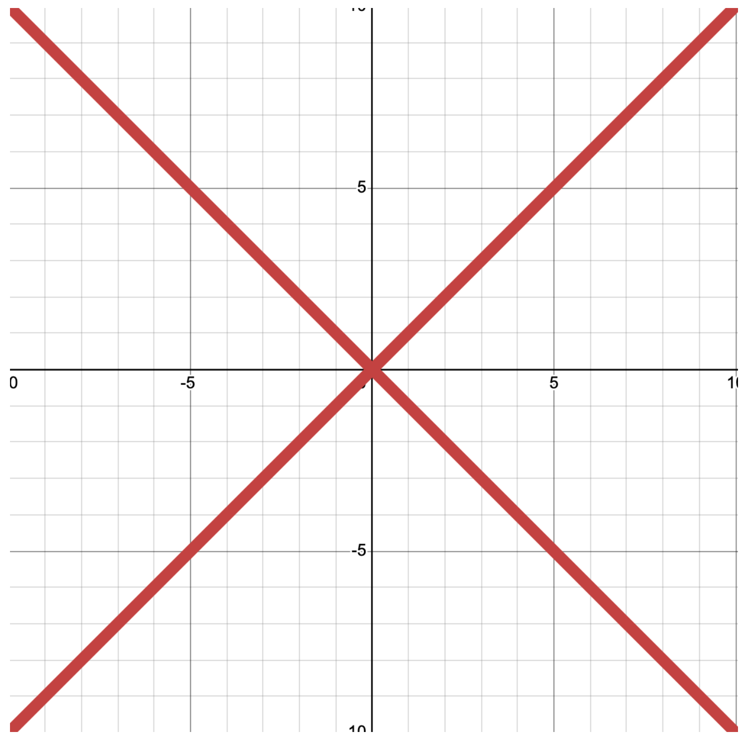
$V(x - y, x^2 + y^2 - z)$

Abductive Inference

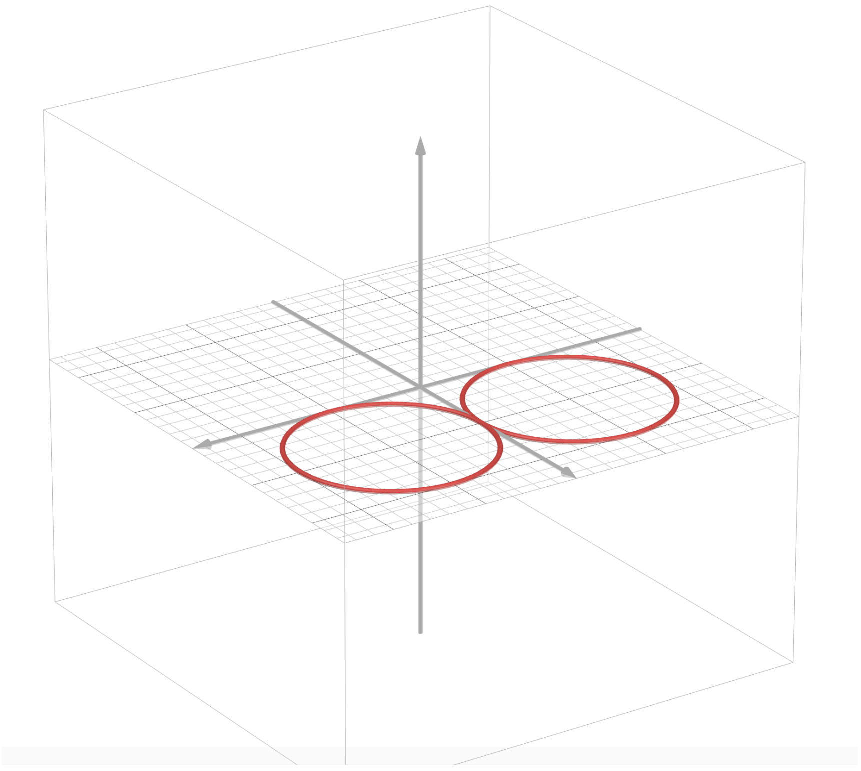
Key idea: Reducibility introduced by Q to the known axioms tells us about residuals.

Irreducible Varieties and Reducible Varieties

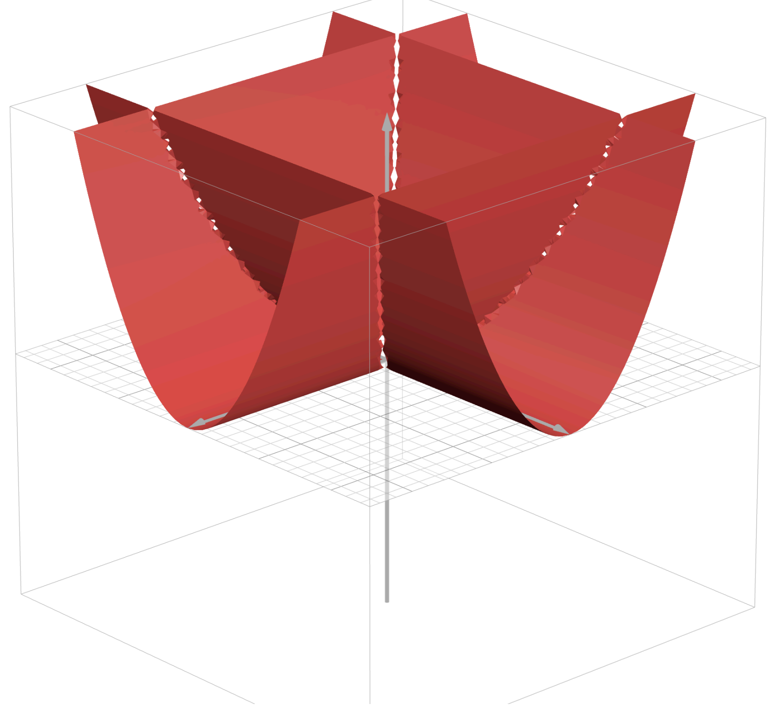
A variety V is irreducible if it cannot be written as the union of two smaller varieties. It is reducible otherwise.



$V(y^2 - x^2)$



$$V(z, ((x-1)^2 + y^2 - 1) \cdot ((x+1)^2 + y^2 - 1))$$



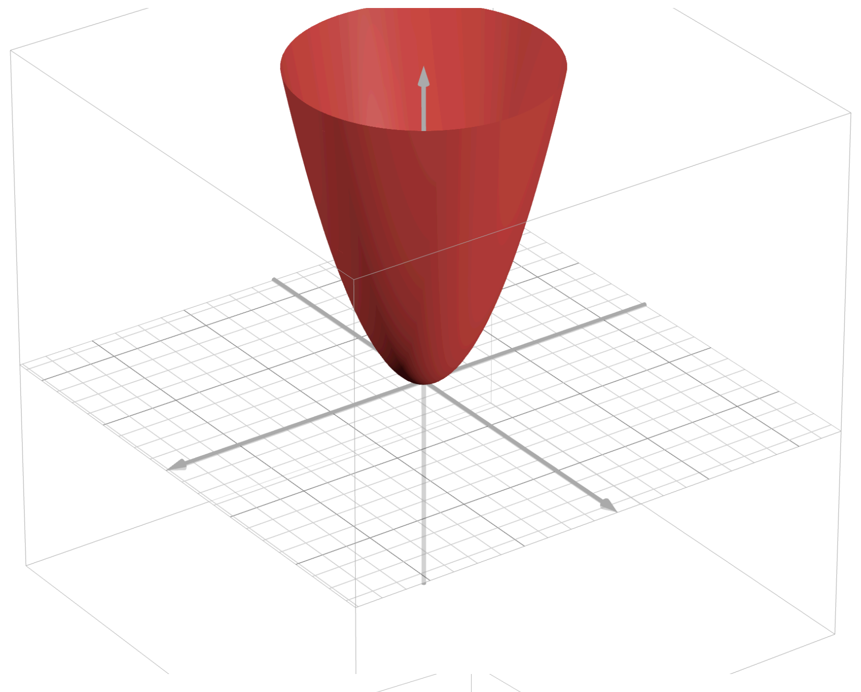
$V((z - x^2)(z - y^2))$

Abductive Inference

Key idea: Reducibility introduced by Q to the known axioms tells us about residuals.

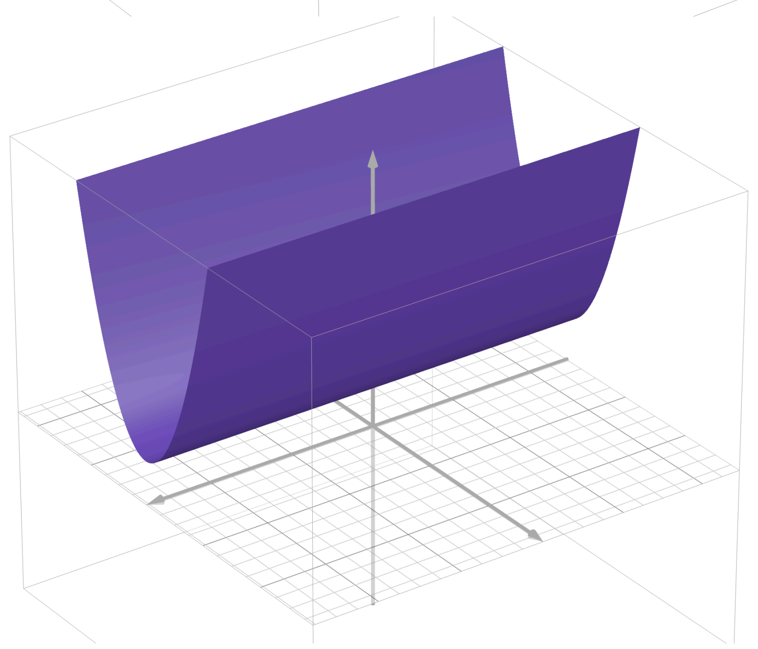
Observation

We can intersect irreducible varieties to obtain reducible varieties.

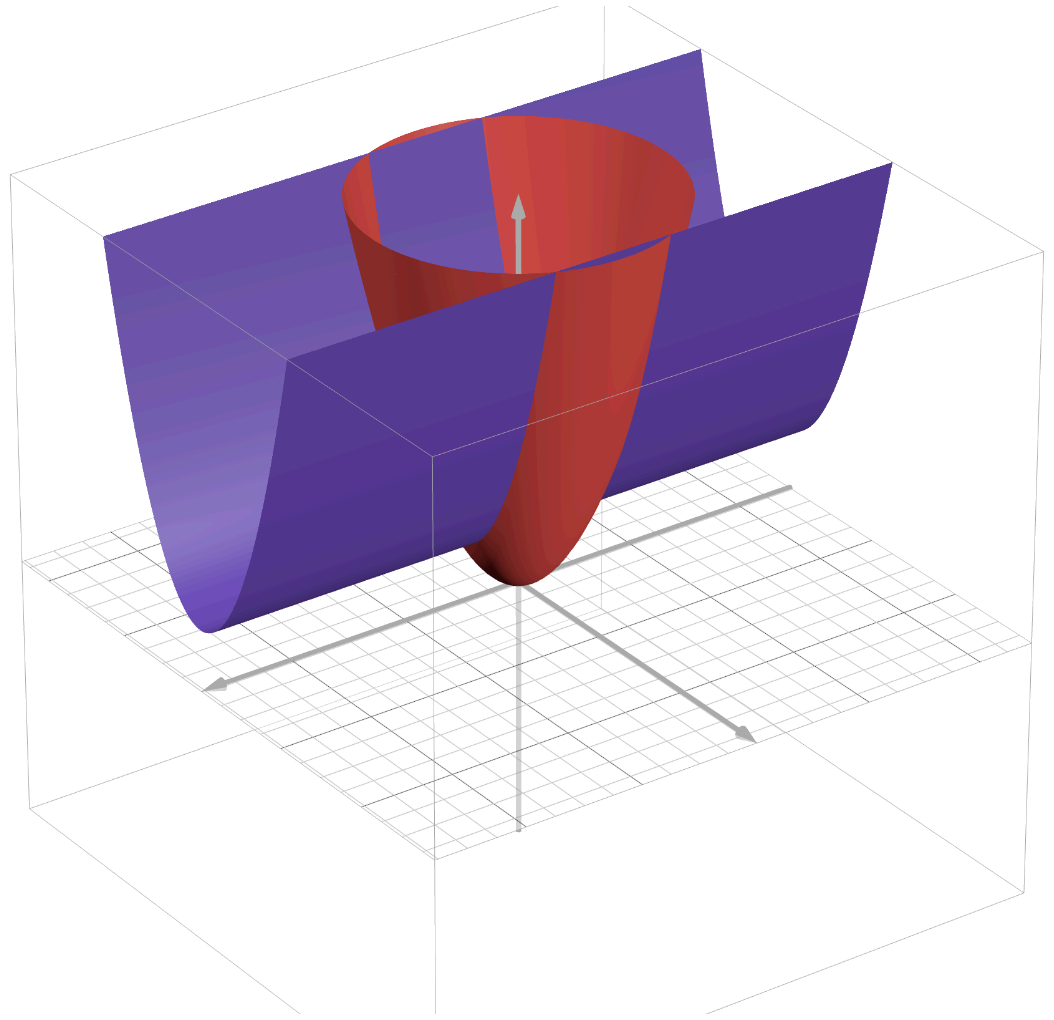


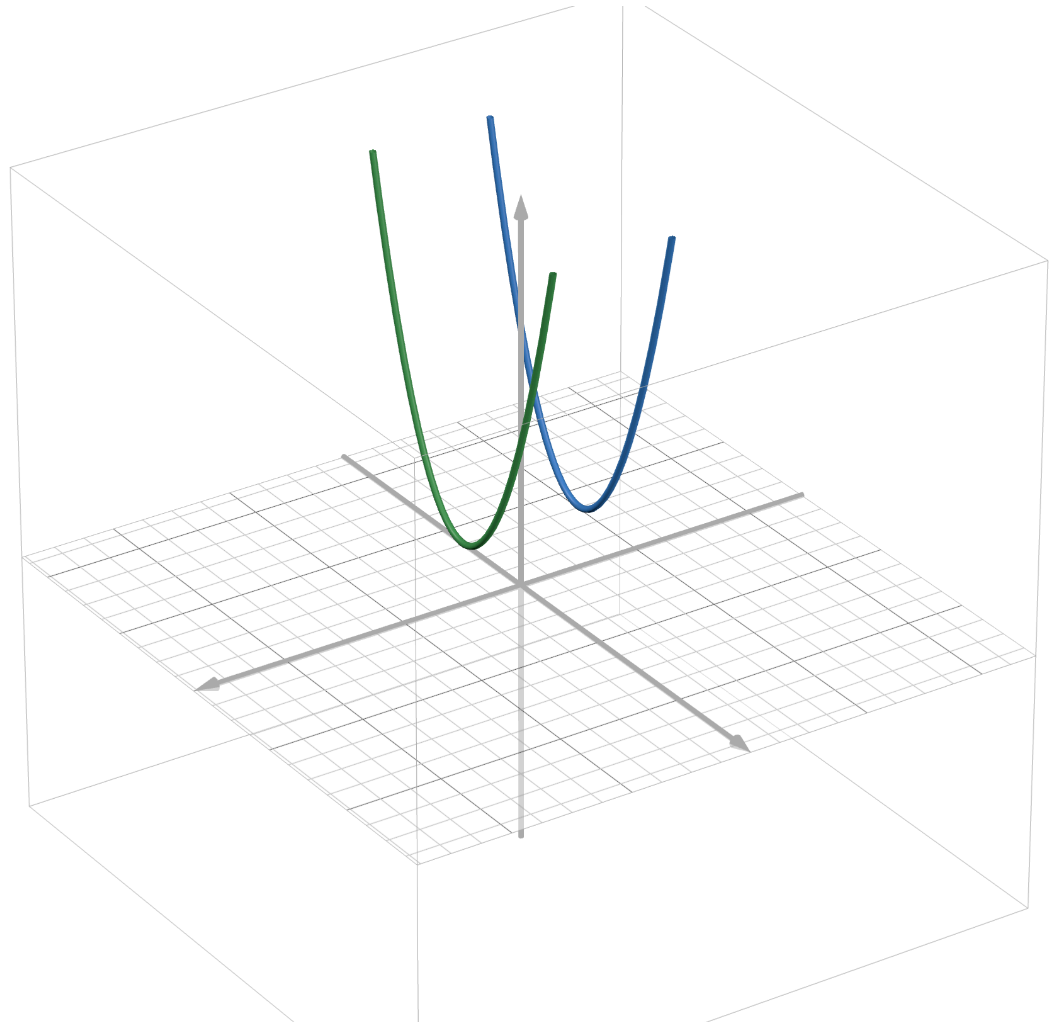
$$V(z - x^2 - y^2)$$

$$\cap$$



$$V(z - y^2 - 1)$$





Abductive Inference

Key idea: Reducibility introduced by Q to the known axioms tells us about residuals.

Observation

We can intersect irreducible varieties to obtain reducible varieties.

Let's assume we have $V(A_1, ..., A_{k-1})$ and $V(Q)$. Then:

$$V(A_1, ..., A_{k-1}) \cap V(Q) = V(A_1, ..., A_{k-1}, Q) = V(\langle A_1, ..., A_{k-1}, Q \rangle)$$

If $Q = \sum_{i=1}^{k} \alpha_i A_i$ for unknown α_i and an unknown A_k, then $\langle A_1, ..., A_{k-1}, Q \rangle = \langle A_1, ..., A_{k-1}, \alpha_k A_k \rangle$.

$$V(A_1, ..., A_{k-1}) \cap V(Q) = V(\langle A_1, ..., A_{k-1}, Q \rangle) = V(\langle A_1, ..., A_{k-1}, \alpha_k A_k \rangle) = V(A_1, ..., A_{k-1}) \cap V(\alpha_k A_k)$$

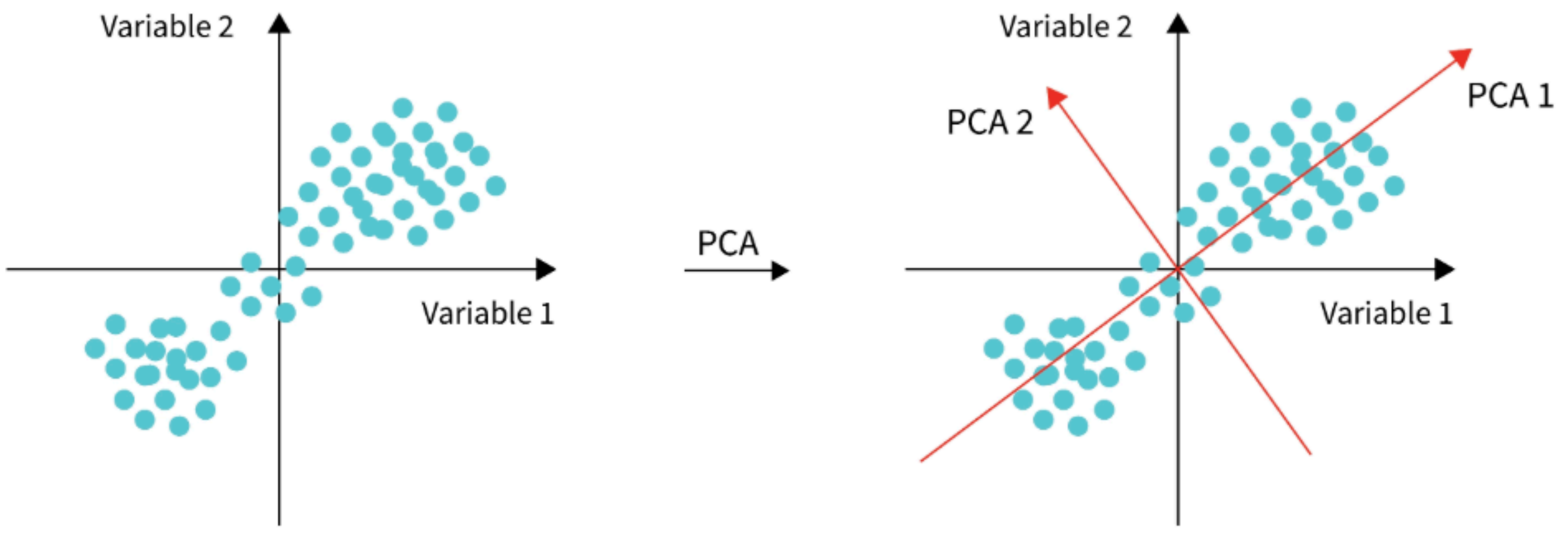
If the residual $\alpha_k A_k$ is non trivial (i.e Q contains some non trivial components in $A_1, ..., A_{k-1}$) then introducing Q introduces reducibility directly related to α_k and A_k.

Abductive Inference

Aim:
Given the axioms $A_1, ..., A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, ..., A_{k-1}, Q)$.

Idea:
Remove the components of Q in the $A_1, ..., A_{k-1}$ directions by looking at the irreducible components of $V(A_1, ..., A_{k-1}, Q)$.

Analogy: PCA



Abductive Inference

Aim:
Given the axioms $A_1, \ldots, A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

Idea:
Remove the components of Q in the $A_1, \ldots, A_{k-1}$ direction by looking at the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

Analogy: PCA

Simplicity criterion: We want candidates for A_k which explain Q that result in minimal residuals with respect to $A_1, \ldots, A_{k-1}$.

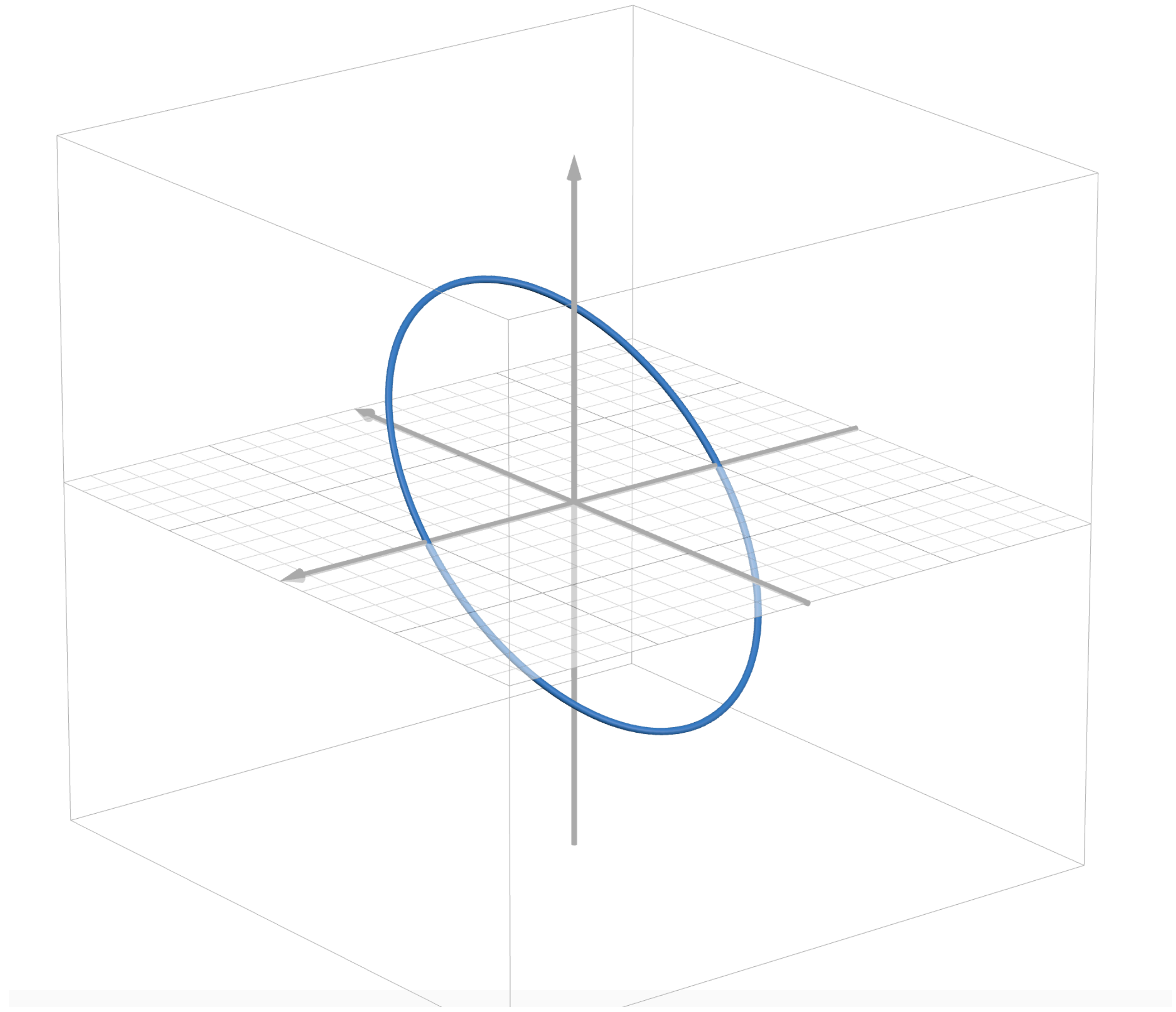
Abductive Inference

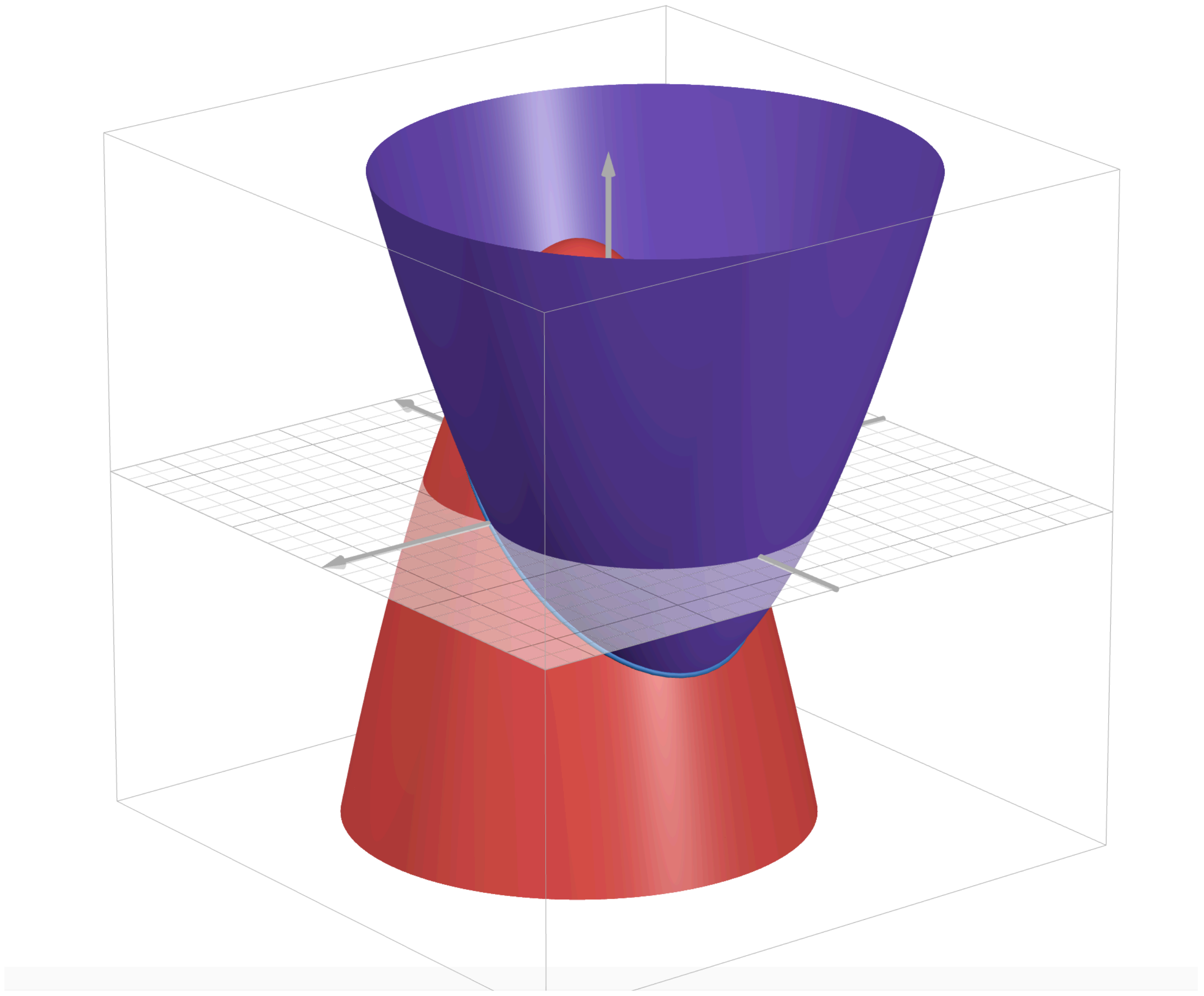
Aim:

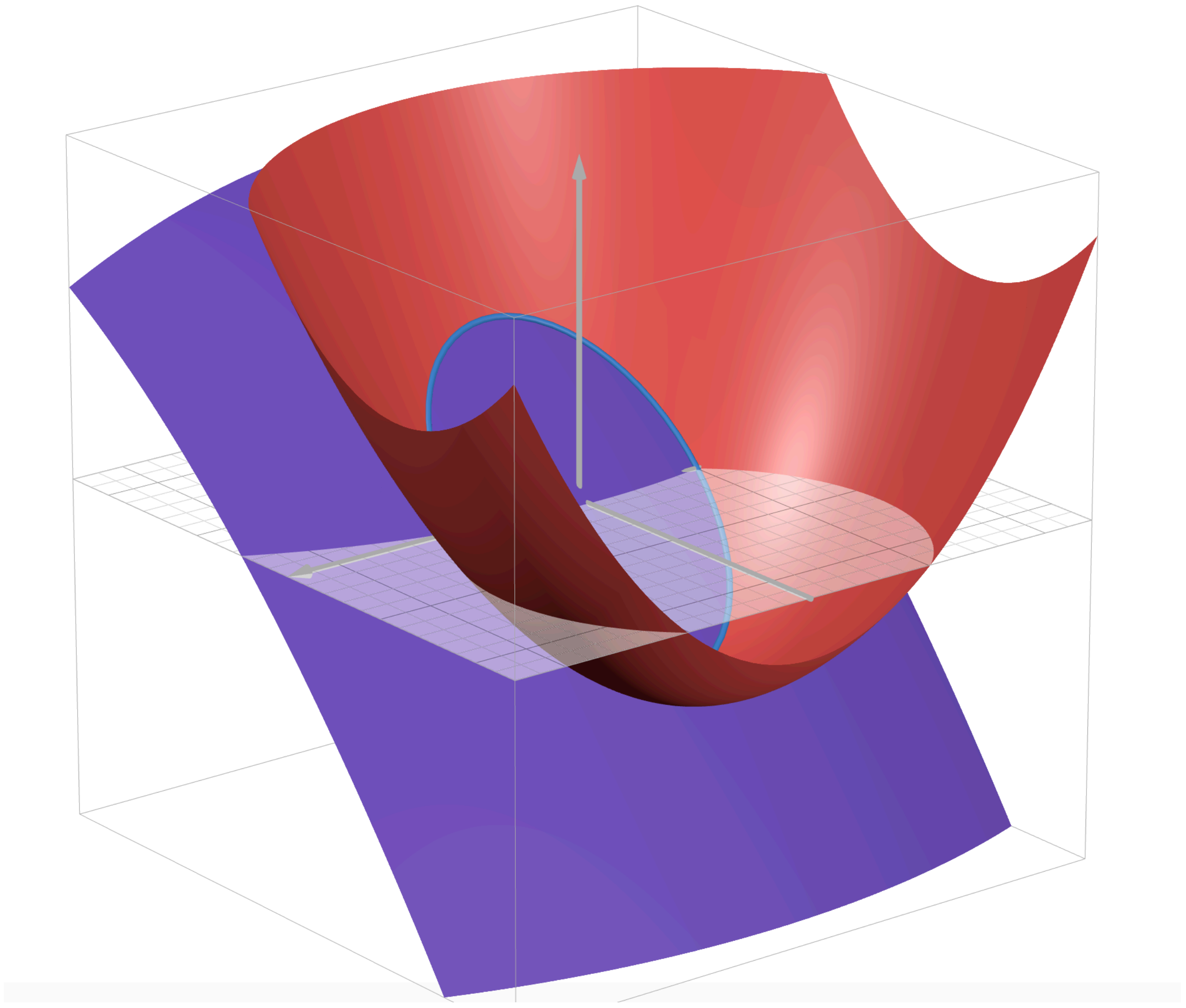
Given the axioms $A_1, \ldots, A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

Concern 1:

There are infinitely many polynomials that could intersect to generate the irreducible component.







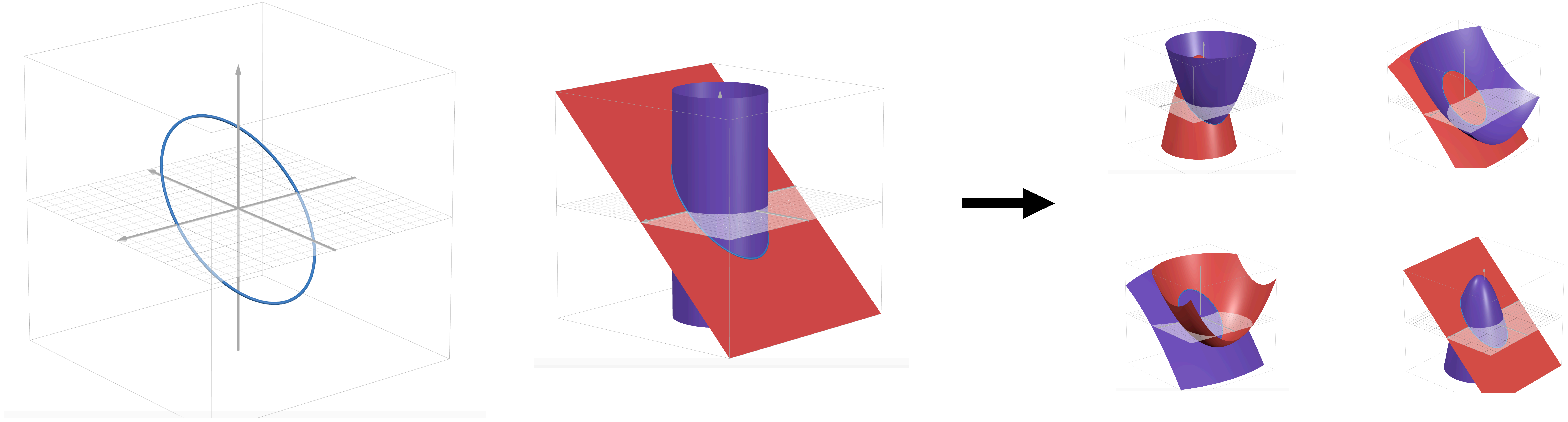
Abductive Inference

Aim:

Given the axioms $A_1, \ldots, A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

Hilbert's Basis Theorem:

Any every ideal I in $\mathbb{R}[\mathbf{x}]$ has a finite set of generators $I = \langle F_1, \ldots, F_r \rangle$



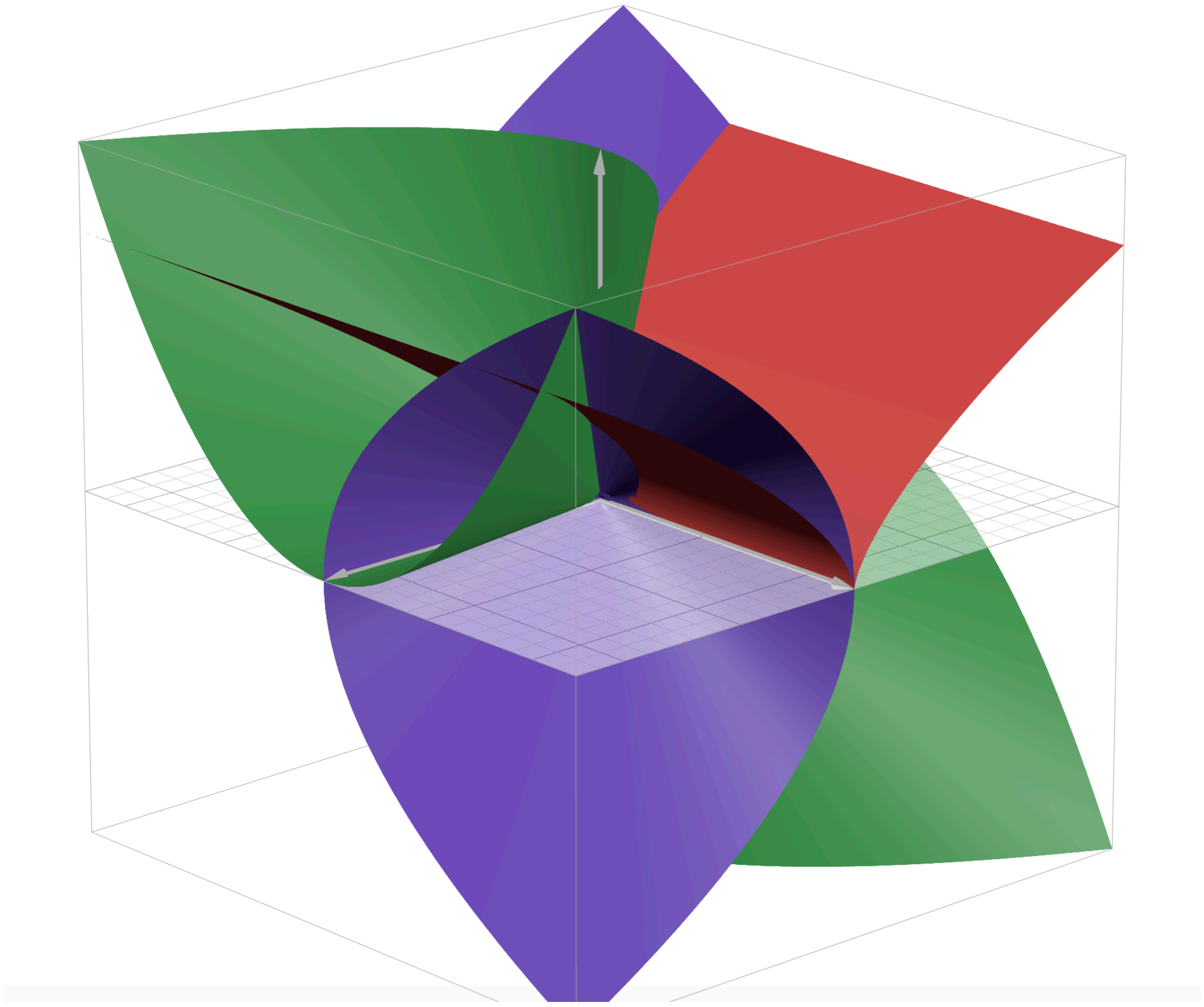
Abductive Inference

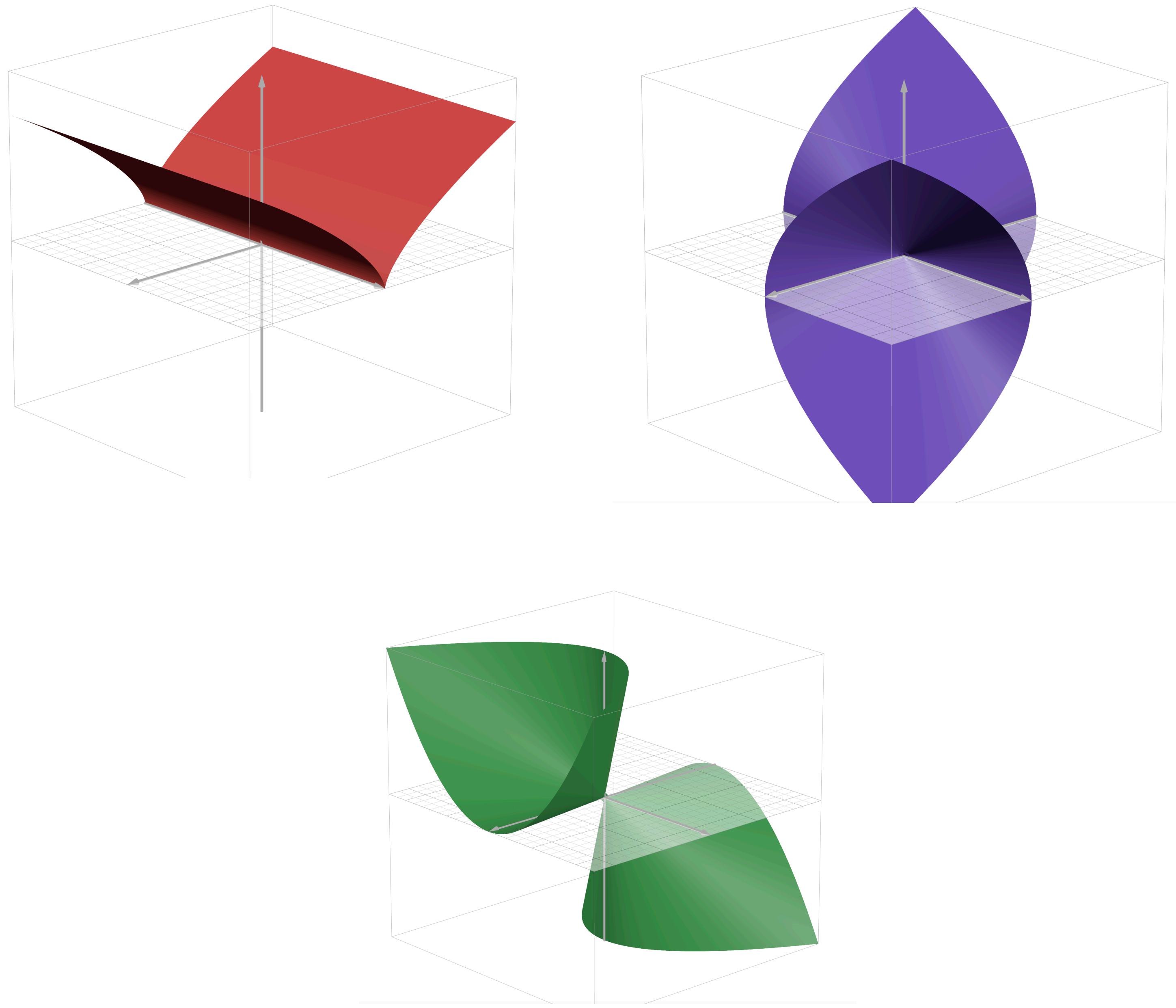
Aim:

Given the axioms $A_1, \ldots, A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

Concern 2:

We need a computational way of finding the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.





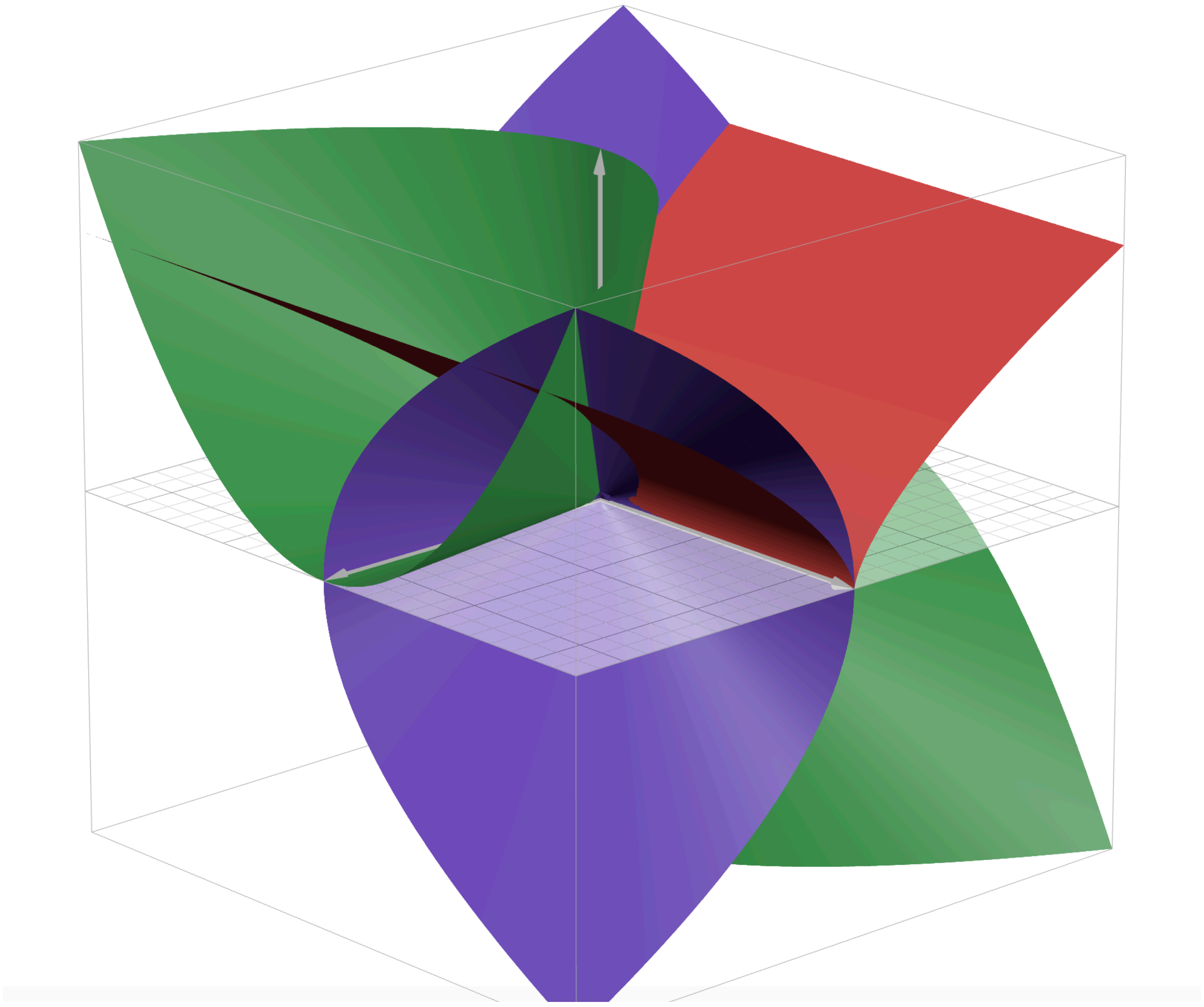
Abductive Inference

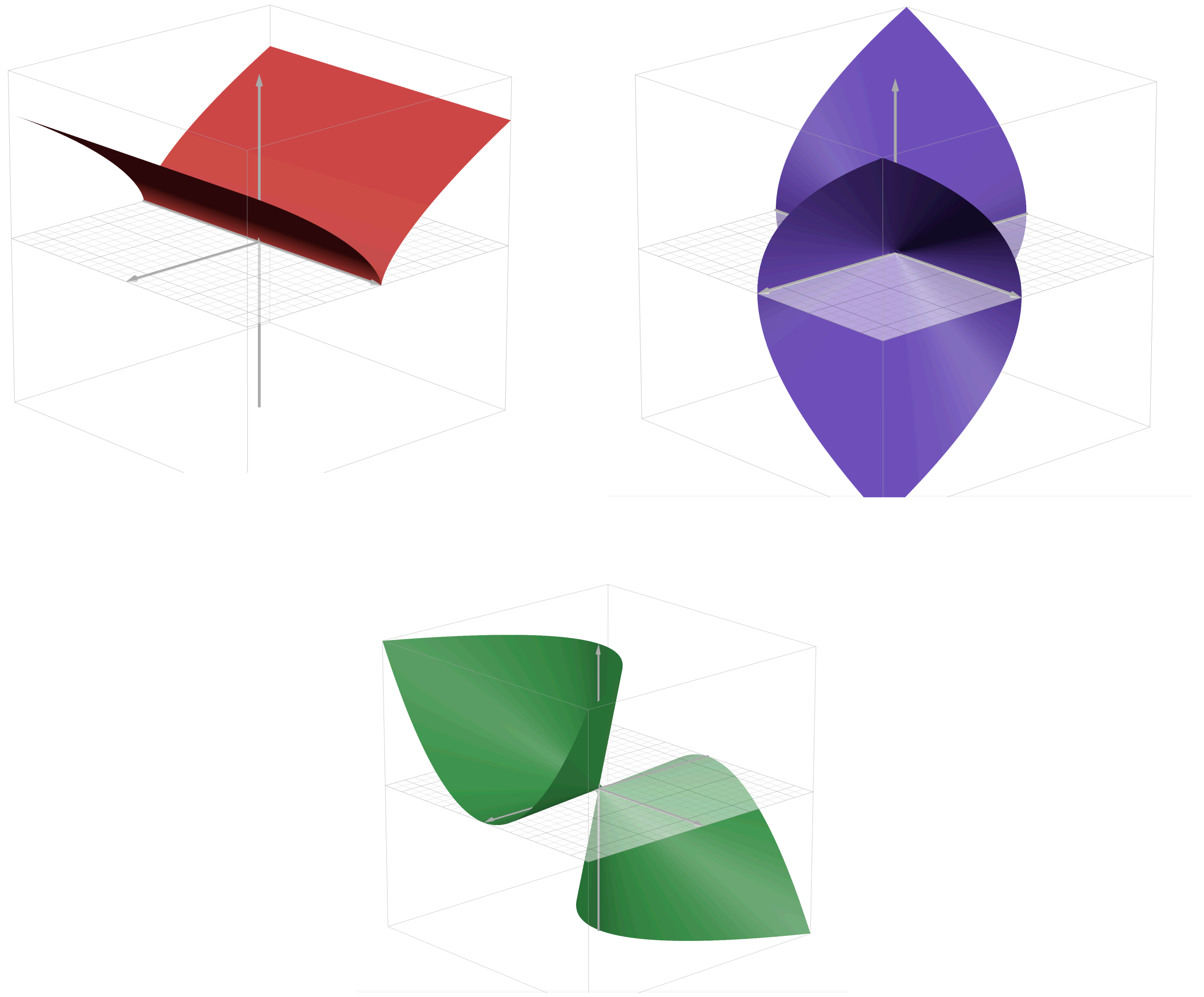
Aim:

Given the axioms $A_1, \ldots, A_{k-1}$ and phenomenon Q, study the irreducible components of $V(A_1, \ldots, A_{k-1}, Q)$.

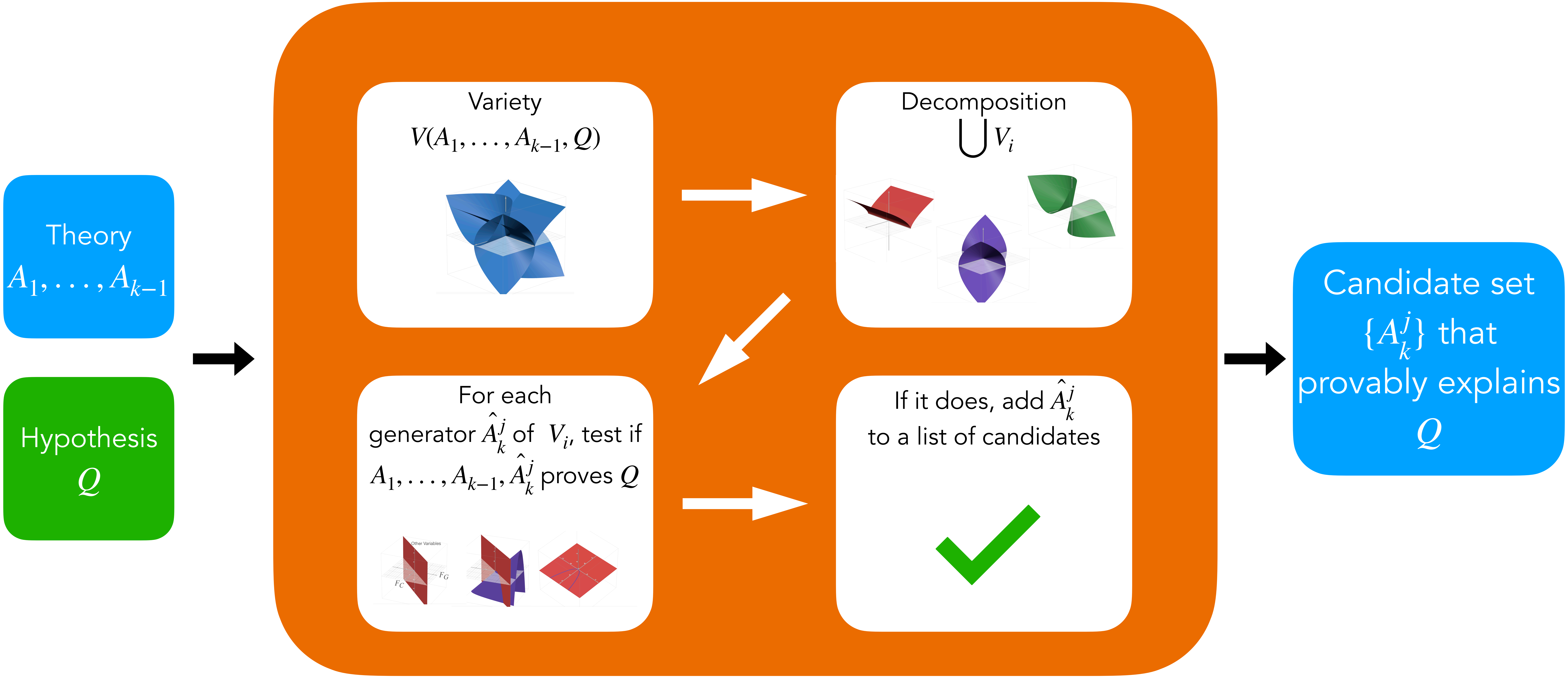
Key Theorem: Lasker-Noether Primary Decomposition Theorem

~ Every variety $V(I)$ can be written as a union of irreducible varieties $\bigcup V_i$





Abductive Inference - Our System



Results - Single Missing Axiom

Kepler's Third Law of Planetary Motion

Axioms

$$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$$
$$F_c - m_2 d_2 w^2 = 0$$
$$F_c - F_g = 0$$
$$wp - 1 = 0$$

$\Longrightarrow$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

Results - Single Missing Axiom

Kepler's Third Law of Planetary Motion

Axioms

$$\sout{(d_1 + d_2)^2 F_g - m_1 m_2 = 0}$$
$$F_c - m_2 d_2 w^2 = 0$$
$$F_c - F_g = 0$$
$$wp - 1 = 0$$

$$\Longrightarrow$$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

We know from AI Feynman and AI Hilbert and other systems that with data, we can still recover Kepler's Law.

Results - Single Missing Axiom

Kepler's Third Law of Planetary Motion

Axioms

~~$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$~~

$F_c - m_2 d_2 w^2 = 0$

$F_c - F_g = 0$

$wp - 1 = 0$

$\implies$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

Computing the primary decomposition gives:

$$\langle m_2, F_g, F_c, wp - 1 \rangle$$

$$\langle F_c - F_g, wp - 1, m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p - w m_2 d_2, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$

This is the only equation in the basis that can be added to the axiom list to derive Kepler.

Results - Single Missing Axiom

Kepler's Third Law of Planetary Motion

Axioms

$$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$$
$$F_c - m_2 d_2 w^2 = 0$$
$$F_c - F_g = 0$$
$$\sout{wp - 1 = 0}$$

$\Longrightarrow$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

Computing the primary decomposition gives:

$$\langle m_2, F_g, F_c \rangle$$
$$\langle d_2, m_1, F_g, F_c \rangle$$
$$\langle m_1, F_c - F_g, (d_1 + d_2)^2, F_g - w^2 m_2 d_2 \rangle$$
$$\langle F_c - F_g, wp - 1, m_1 p^2 - d_1^2 d_2 - 2d_1 d_2^2 - d_2^3, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$
$$\langle F_c - F_g, wp + 1, m_1 p^2 - d_1^2 d_2 - 2d_1 d_2^2 - d_2^3, F_g p^2 + m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$

Results - Single Missing Axiom

Kepler's Third Law of Planetary Motion

Axioms

$$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$$
$$F_c - m_2 d_2 w^2 = 0$$
$$F_c - F_g = 0$$
$$\sout{wp - 1 = 0}$$

$\Longrightarrow$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

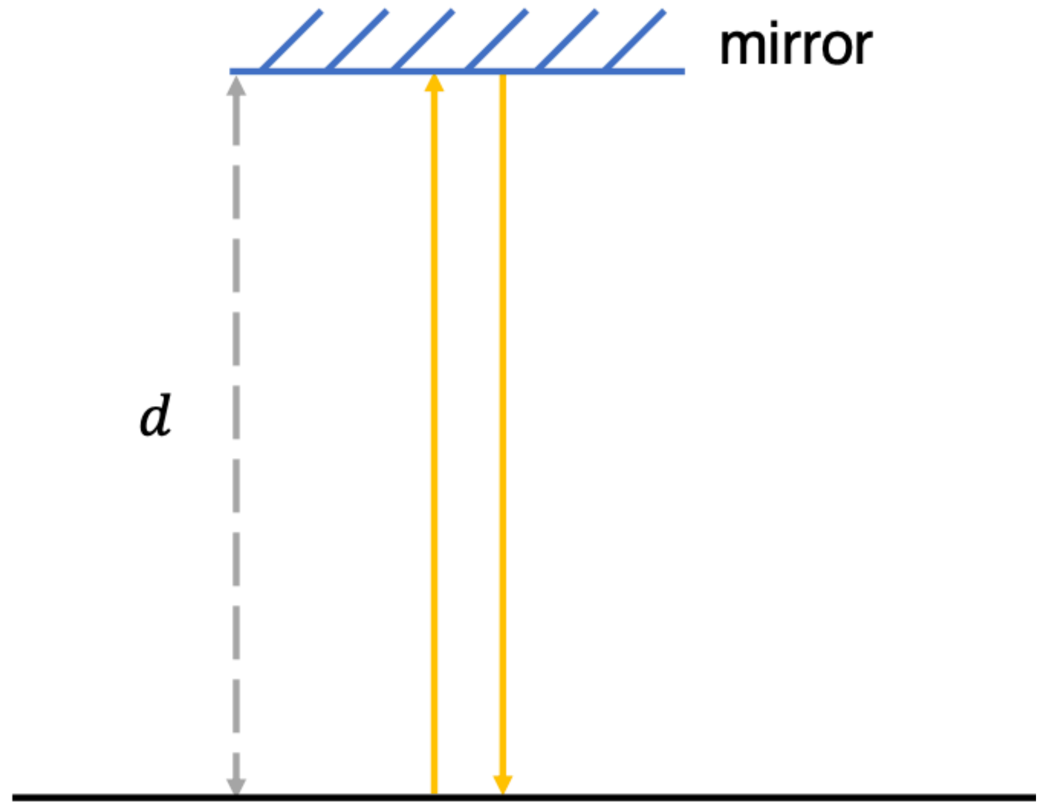
Looking at the last two components, which contain polynomials that can be used to derive Kepler:

$$\langle F_c - F_g, wp - 1, m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$
$$\langle F_c - F_g, wp + 1, m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p^2 + m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$

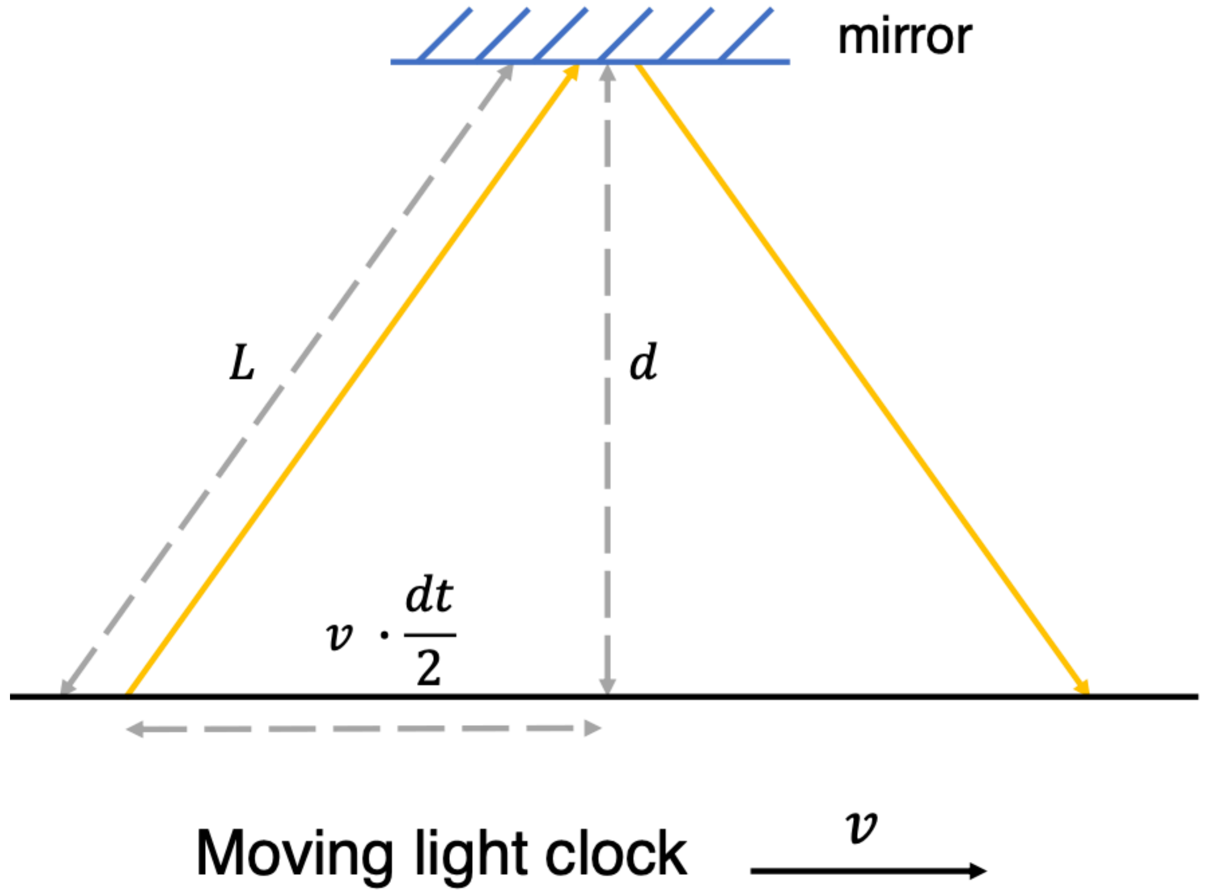
Why $wp + 1$? Because in the algebraic combination $Q = \sum_{i=1}^{4} \alpha_i A_i$, it turns out $\alpha_4 = wp + 1$.

Results - Single Missing Axiom

Einstein’s Relativistic Time Dilation Law



Stationary light clock



Moving light clock

Correct Axioms

$$cdt_0 - 2d = 0$$
$$cdt - 2L = 0$$
$$L^2 = d^2 + v(dt/2)^2$$
$$f_0 = 1/dt_0$$
$$f = 1/dt$$

$\Longrightarrow$

Phenomenon

$$\frac{f - f_0}{f} = \sqrt{1 - \frac{v^2}{c^2}} - 1$$

Incorrect Axioms

$$cdt_0 - 2d = 0$$
$$dt = 2L/\sqrt{v^2 + c^2}$$
$$L^2 = d^2 + v(dt/2)^2$$
$$f_0 = 1/dt_0$$
$$f = 1/dt$$

$\Longrightarrow$

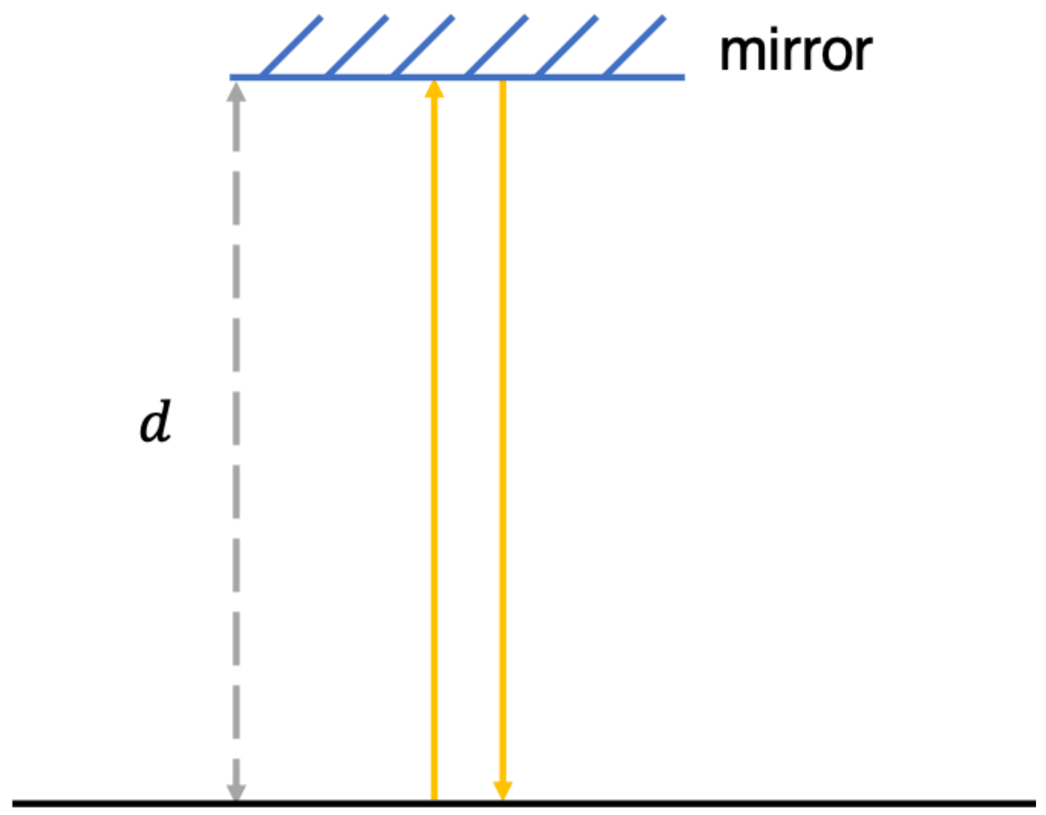
Phenomenon

$$\frac{f - f_0}{f} = \sqrt{1 - \frac{v^2}{c^2}} - 1$$

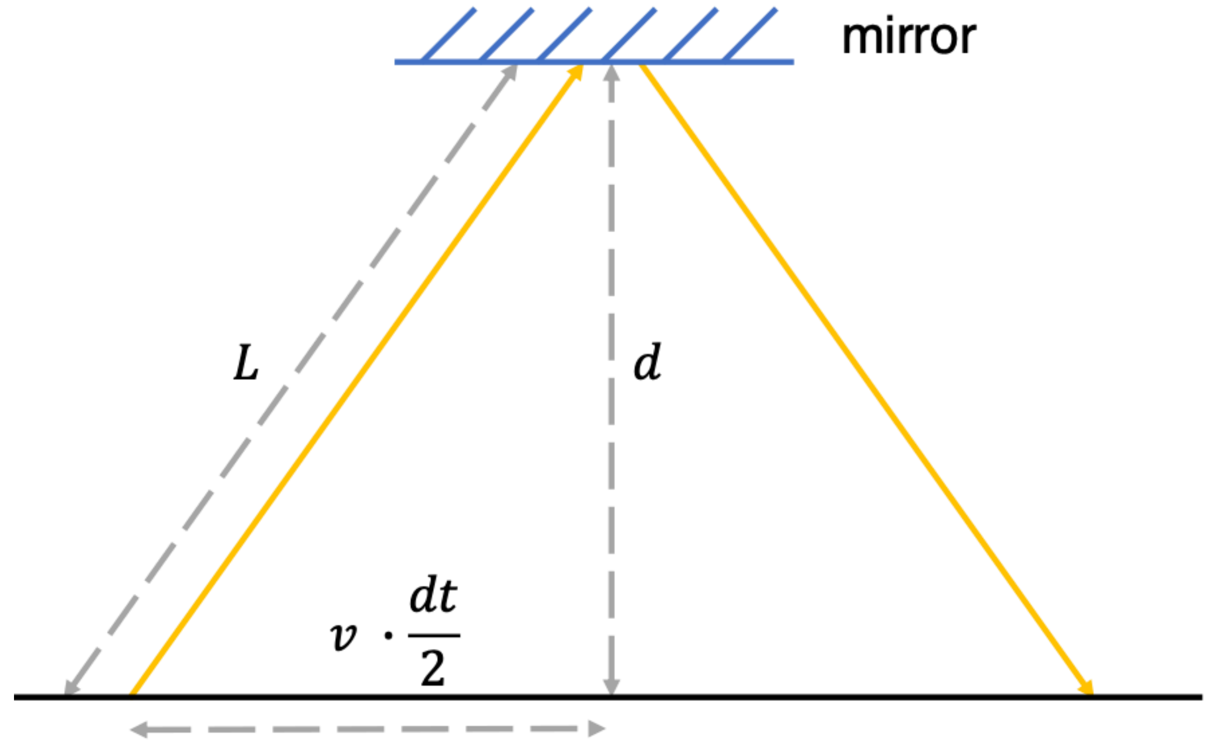
We know that both AI Descartes and AI Hilbert discover the correct formula regardless of theory, and can select the correct axiom.

Results - Single Missing Axiom

Einstein's Relativistic Time Dilation Law



Stationary light clock



Moving light clock v

Correct Axioms

$$cdt_0 - 2d = 0$$
~~$cdt - 2L = 0$~~
$$L^2 = d^2 + v(dt/2)^2$$
$$f_0 = 1/dt_0$$
$$f = 1/dt$$

$\Longrightarrow$

Phenomenon

$$\frac{f - f_0}{f} = \sqrt{1 - \frac{v^2}{c^2}} - 1$$

Incorrect Axioms

$$cdt_0 - 2d = 0$$
~~$dt = 2L/\sqrt{v^2 + c^2}$~~
$$L^2 = d^2 + v(dt/2)^2$$
$$f_0 = 1/dt_0$$
$$f = 1/dt$$

$\Longrightarrow$

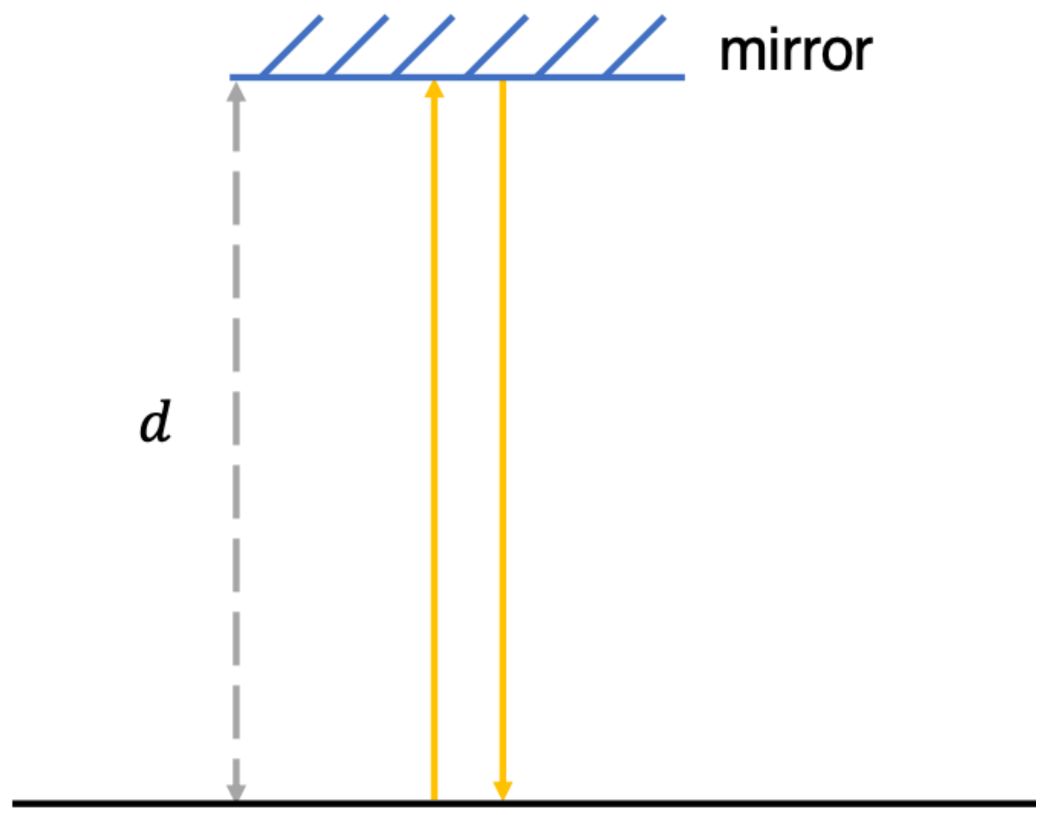
Phenomenon

$$\frac{f - f_0}{f} = \sqrt{1 - \frac{v^2}{c^2}} - 1$$

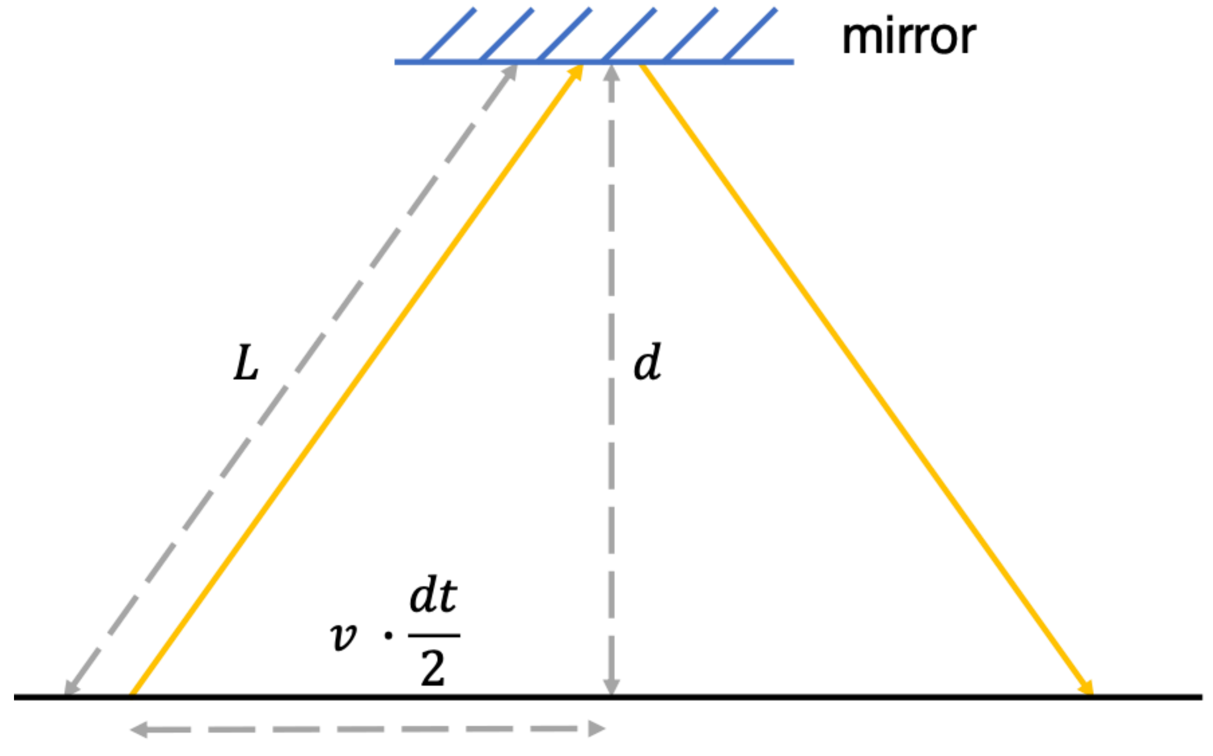
Even if the system is incomplete, both systems can correctly recover the law.

Results - Single Missing Axiom

Einstein's Relativistic Time Dilation Law



Stationary light clock



Moving light clock v

Incorrect Axioms

$cdt_0 - 2d = 0$

~~$cdt - 2L = 0$~~

$L^2 = d^2 + v(dt/2)^2$

$f_0 = 1/dt_0$

$f = 1/dt$

$\Longrightarrow$

Phenomenon

$$\frac{f - f_0}{f} = \sqrt{1 - \frac{v^2}{c^2}} - 1$$

Out system generates the following candidates for the missing axiom:

$$cdt - 2L = 0$$

$$2Lf + c = 0$$

If we had no information about the speed of light, we recover that the speed of light must be constant.

Results - Single Missing Axiom

Problem	# Axioms Recovered	Avg. Time (s)	Total Axioms
Kepler	4/4	0.1	4
Compton	10/10	5.4	10
Einstein	5/5	1.5	5
Escape Velocity	5/5	0.4	5
Light Damping	5/5	1.6	5
Hagen Poiseuille	4/4	0.6	4
Neutrino Decay	5/5	3.5	5
Hall Effect	7/9	11.1	9
Carrier-Resolved PhotoHall Effect	7/7	1.1	7

Results - Multiple Missing Axioms

Axioms

$$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$$
$$F_c - m_2 d_2 w^2 = 0$$
~~$F_c - F_g = 0$~~
~~$wp - 1 = 0$~~

$\Longrightarrow$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

What if we're missing more axioms or need to make more corrections?

$$\langle m_2, F_c, (d_1 + d_2)^2 \rangle$$
$$\langle m_1, (d_1 + d_2)^2, Fc - w^2 m_2 d_2 \rangle$$
$$\langle m_2, F_g, F_c \rangle$$
$$\langle m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2, F_c - w^2 m_2 d_2 \rangle$$

We don't seem to recover the two missing axioms. Question: What are the criteria for derivability?

Results - Condition for Derivibility

Theorem:

For ideals I, J, if $\dim I = \dim J$ and $I \subseteq J$, then $V(I)$ and $V(J)$ share a common irreducible component.

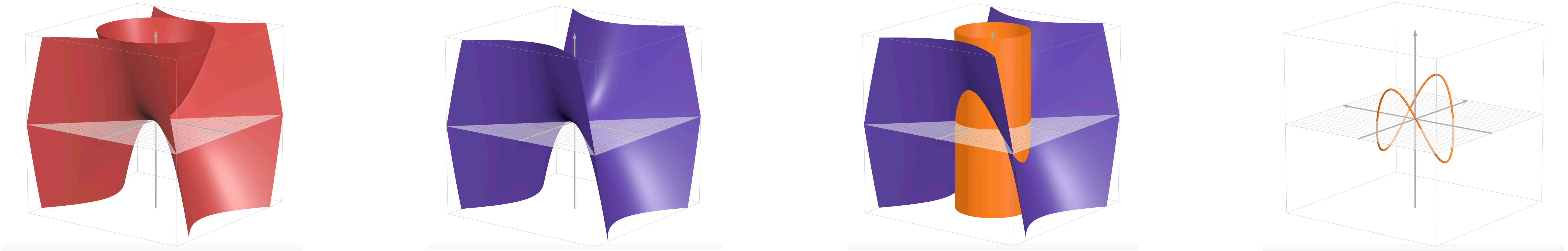
Missing r axioms / requiring r corrections to explain j phenomena:

Take $I = \langle A_1, \ldots, A_{k-r}, Q_1, \ldots, Q_j \rangle$

Take $J = \langle A_1, \ldots, A_k \rangle$

Then we can recover $A_{k-r+1}, \ldots, A_k$ from the shared irreducible component if $\dim I = \dim J$.

Main idea: Intersecting a polynomial surface $V(A_i)$ with a variety $V(A_1, \ldots, A_{i-1})$ decreases the dimension by at most one. ***Any dimensions of information lost in** $A_{k-r+1}, \ldots, A_k$ **need to be made up by** $Q_1, \ldots, Q_j$ **to guarantee recovery.***



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This is in line with what we had before:

If $Q_1 = \sum_{i=1}^{k} \alpha_i A_i$, $Q_2 = \sum_{i=1}^{k} \beta_i A_i$, then for unknown $\alpha_i, \beta_j, A_{k-1}, A_k$, then

$$\langle A_1, \ldots A_{k-2}, Q_1, Q_2 \rangle = \langle A_1, \ldots A_{k-2}, \gamma_{k-1} A_{k-1}, \gamma_k A_k \rangle$$

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~~$F_c - F_g = 0$~~

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$$\Longrightarrow$$

Phenomenon

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But we still recover some information!

$$\langle m_2, F_c, (d_1 + d_2)^2 \rangle$$
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Phenomenon

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But we still recover some information!

$$\langle m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2, F_c - w^2 m_2 d_2 \rangle$$

Axiom 2: $F_c - m_2 d_2 w^2 = 0$
Inferred : $F_g p^2 - m_2 d_2 = 0$

We have discovered a reformulation of centrifugal force with $F_c = F_g, w = 1/p$ swapped in.

Results - Multiple Missing Axioms

Axioms

$(d_1 + d_2)^2 F_g - m_1 m_2 = 0$

~~$F_c - m_2 d_2 w^2 = 0$~~

~~$F_c - F_g = 0$~~

~~$wp - 1 = 0$~~

$\Longrightarrow$

Phenomenon

$$p = \sqrt{\frac{4(d_1 + d_2)^3}{(m_1 + m_2)}}$$

One step further: Missing 3 out of 4 axioms.

$$\langle m_1 p^2 - d_1^2 d_2 - 2 d_1 d_2^2 - d_2^3, F_g p^2 - m_2 d_2, F_g (d_1 + d_2)^2 - m_1 m_2 \rangle$$

Inferred: $F_g p^2 - m_2 d_2 = 0$ | This time, without knowing anything about centrifugal force, we recover the centrifugal force equation with $F_c = F_g, w = 1/p$ swapped in

Results - Multiple Missing Axioms

Problem	Missing Axioms (Tuple)	CPU Time	Recovered
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ F_c - m_2 d_2 w^2\}$	0.1s	X
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ F_c - F_g\}$	0.3s	✓
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ wp - 1\}$	0.1s	X
Kepler	$\{F_c - m_2 d_2 w^2,\ F_c - F_g\}$	0.1s	✓
Kepler	$\{F_c - m_2 d_2 w^2,\ wp - 1\}$	0.2s	✓
Kepler	$\{F_c - F_g,\ wp - 1\}$	0.1s	✓
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ F_c - m_2 d_2 w^2,\ F_c - F_g\}$	0.1s	X
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ F_c - m_2 d_2 w^2,\ wp - 1\}$	0.1s	X
Kepler	$\{(d_1 + d_2)^2 F_g - m_1 m_2,\ F_c - F_g,\ wp - 1\}$	0.1s	X
Kepler	$\{F_c - m_2 d_2 w^2,\ F_c - F_g,\ wp - 1\}$	0.2s	✓

Results - Multiple Missing Axioms

Problem	Missing Axioms (Tuple)	Time	Recovered
Einstein	$\{cdt_0 - 2d,\ 4L^2 - 4d^2 - v^2dt^2\}$	0.4s	✓
Einstein	$\{cdt_0 - 2d,\ f_0dt_0 - 1\}$	0.1s	✓
Einstein	$\{cdt_0 - 2d,\ fdt - 1\}$	0.1s	X
Einstein	$\{cdt_0 - 2d,\ cdt - 2L\}$	0.1S	✓
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ f_0dt_0 - 1\}$	0.2s	X
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ fdt - 1\}$	0.1s	✓
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ cdt - 2L\}$	0.1s	✓
Einstein	$\{f_0dt_0 - 1,\ fdt - 1\}$	0.2s	X
Einstein	$\{f_0dt_0 - 1,\ cdt - 2L\}$	0.1s	X
Einstein	$\{fdt - 1,\ cdt - 2L\}$	0.1s	✓
Einstein	$\{cdt_0 - 2d,\ 4L^2 - 4d^2 - v^2dt^2,\ f_0dt_0 - 1\}$	0.1s	X
Einstein	$\{cdt_0 - 2d,\ 4L^2 - 4d^2 - v^2dt^2,\ fdt - 1\}$	0.1s	X
Einstein	$\{cdt_0 - 2d,\ 4L^2 - 4d^2 - v^2dt^2,\ cdt - 2L\}$	0.3s	✓
Einstein	$\{cdt_0 - 2d,\ f_0dt_0 - 1,\ fdt - 1\}$	0.1s	X
Einstein	$\{cdt_0 - 2d,\ f_0dt_0 - 1,\ cdt - 2L\}$	0.1s	X
Einstein	$\{cdt_0 - 2d,\ fdt - 1,\ cdt - 2L\}$	0.2s	X
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ f_0dt_0 - 1,\ fdt - 1\}$	0.1s	X
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ f_0dt_0 - 1,\ cdt - 2L\}$	0.1s	✓
Einstein	$\{4L^2 - 4d^2 - v^2dt^2,\ fdt - 1,\ cdt - 2L\}$	0.1s	X
Einstein	$\{f_0dt_0 - 1,\ fdt - 1,\ cdt - 2L\}$	0.1s	X

Results - Multiple Missing Axioms

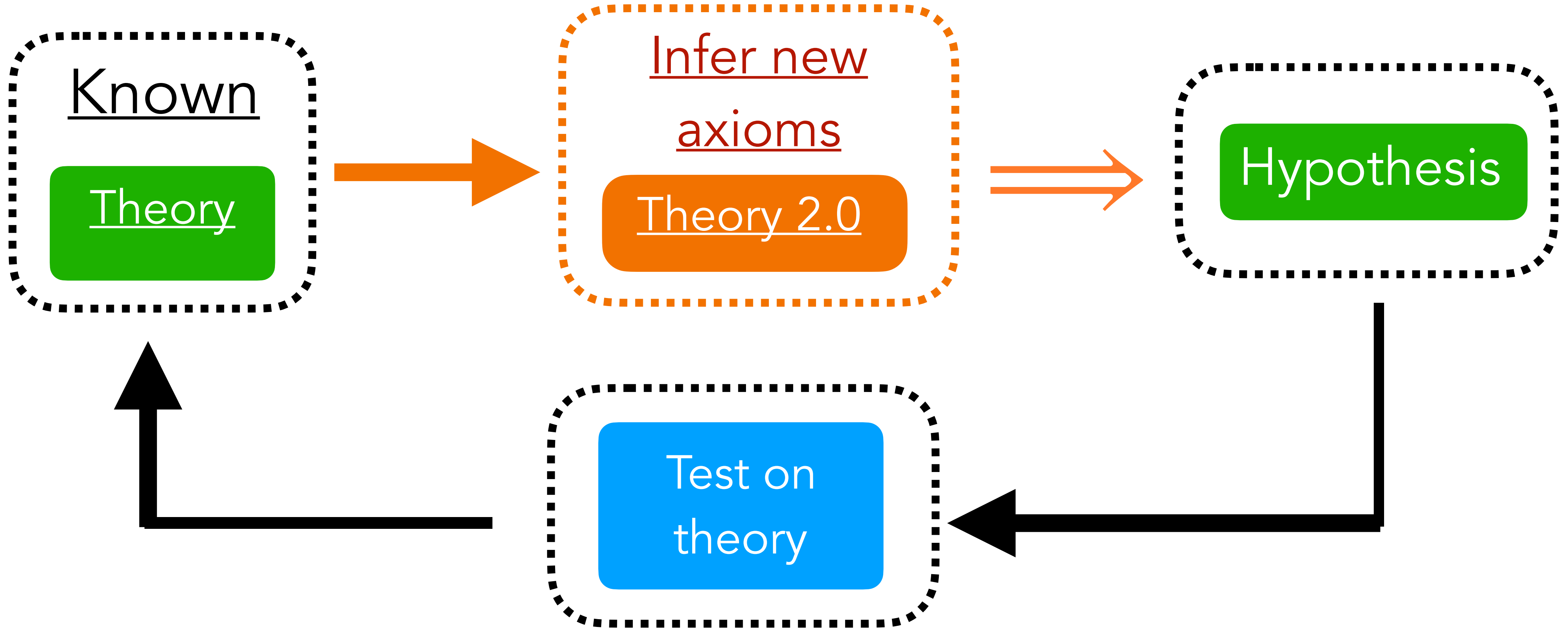
Problem	# Tuples Recovered	Avg. Time (s)	# of Axioms
Kepler	5/10	0.1	4
Einstein	8/20	1.5	5
Escape Velocity	6/20	0.4	5
Light Damping	5/20	1.6	5
Hagen Poiseuille	6/10	0.6	4
Neutrino Decay	7/20	3.5	5

Limitations, Ongoing, and Future Work

1. Polynomials: We are currently restricted to polynomials (including traditional polynomials, trig polynomials, etc). We cannot handle ODEs / PDEs (yet).

2. Sensitive to noise: This method is somewhat sensitive to noisy in coefficients of phenomena polynomials due to the exact nature of computer algebra computations.

Given that there's more work to be done, we believe this is a step in the right direction of augmenting the scientific method using modern tooling.



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